Using Powerful Prior Knowledge of Diffusion Model in Deep Unfolding Network for Image Compressive Sensing

Supplementary Material

1. Proof of Theorem 1

Let's write the assumptions again here:

Assumption 1. *The Gaussian sensing matrix* $\mathbf{\Phi} \sim \mathcal{N}(0, \mathbf{I}/M)$. *The sensing rate* $\delta = M/N \in \mathbb{R}(0,1)$, *where* $M, N \to +\infty$. *The denoising function* $\eta_t(\cdot)$ *is Lipschitz continuous, with* $t \in \mathbb{N}$.

Assumption 2. *The neural network* $p_\theta(\mathbf{x}_{t-1} \,|\, \mathbf{x}_t)$ *is a perfect denoiser, meaning* $p_\theta(\mathbf{x}_{t-1} \,|\, \mathbf{x}_t) \equiv q(\mathbf{x}_{t-1} \,|\, \mathbf{x}_t, \mathbf{x}_0)$. *Consequently,* $\mathbf{x}_t \sim \mathcal{N}(\sqrt{\bar{\alpha}_t}\,\mathbf{x}, (1-\bar{\alpha}_t)\,\mathbf{I})$ *for all* $t \in \mathbb{N}$.

Assumption 3. *A perfect Gaussian filter exists and is represented by* $\mathcal{D}_t(\cdot)$. *The distribution* $q(\mathcal{D}_t(\mathbf{d}_t)|\,\mathbf{d}_t) = \mathcal{N}(\sqrt{\bar{\alpha}_t}\,\mathbf{x}, (1-\sqrt{\bar{\alpha}_t})\,\mathbf{I})$, *where* $\mathbf{d}_t \sim \mathcal{N}(\sqrt{\bar{\alpha}_t}\,\mathbf{x}, (1-\sqrt{\bar{\alpha}_t})\,\mathbf{I}/\delta)$.

AMP gives the following iterative formula to reconstruct the image with the iteration starts from $t = +\infty$ and ends when $t = 0$, and the final answer $\mathbf{x}_0$ will be converged to ground truth:

$$\begin{aligned}
\mathbf{u}_t &= \mathbf{y} - \mathbf{\Phi}\mathbf{x}_t + \mathbf{u}_{t+1}\,\mathrm{div}\eta_t(\mathbf{h}_{t+1})/M \\
\mathbf{h}_t &= \mathbf{\Phi}^{\mathrm{T}}\mathbf{u}_t + \mathbf{x}_t \\
\mathbf{x}_{t-1} &= \eta_t(\mathbf{h}_t)
\end{aligned} \tag{1}$$

and the iterative formula approximately follows the following state evolution:

$$\begin{aligned}
\mathbf{h}_t &\sim \mathcal{N}(\mathbf{x}, \sigma_t^2\,\mathbf{I}/\delta) \\
\sigma_{t-1}^2 &= \mathbb{E}\left\{[\eta_t(\mathbf{h}_t) - \mathbf{x}]^2\right\}
\end{aligned} \tag{2}$$

Let $o_t(\mathbf{u}_{t+1}, \mathbf{h}_{t+1}) := \mathbf{\Phi}^{\mathrm{T}}\,\mathbf{u}_{t+1}\,\mathrm{div}\eta_t(\mathbf{h}_{t+1})$, and the AMP iterative formula in Eq. (1) can be rewritten by the following formula:

$$\begin{aligned}
\mathbf{s}_t &= \mathbf{x}_t - \mathbf{\Phi}^{\mathrm{T}}(\mathbf{\Phi}\mathbf{x}_t - \mathbf{y}) \\
\mathbf{h}_t &= \mathbf{s}_t + o_t(\mathbf{u}_{t+1}, \mathbf{h}_{t+1}) \\
\mathbf{x}_{t-1} &= \eta_t(\mathbf{h}_t)
\end{aligned} \tag{3}$$

To avoid any confusion of symbols, herein we rewrite Eq. (3) as follows:

$$\begin{aligned}
\tilde{\mathbf{s}}_t &= \tilde{\mathbf{x}}_t - \mathbf{\Phi}^{\mathrm{T}}(\mathbf{\Phi}\,\tilde{\mathbf{x}}_t - \mathbf{y}) \\
\mathbf{h}_t &= \tilde{\mathbf{s}}_t + o_t(\mathbf{u}_{t+1}, \mathbf{h}_{t+1}) \\
\tilde{\mathbf{x}}_{t-1} &= \eta_t(\mathbf{h}_t)
\end{aligned} \tag{4}$$

where the state evolution of the above formula is as follows:

$$\begin{aligned}
\mathbf{h}_t &\sim \mathcal{N}(\mathbf{x}, \tilde{\sigma}_t^2\,\mathbf{I}/\delta) \\
\tilde{\sigma}_{t-1}^2 &= \mathbb{E}\left\{[\eta_t(\tilde{\mathbf{h}}_t) - \mathbf{x}]^2\right\}
\end{aligned} \tag{5}$$

We define $\eta_t(\mathbf{h}_t) := p_\theta(\mathbf{x}_{t-1}\,|\mathcal{D}_t(\sqrt{\bar{\alpha}_t}\,\mathbf{h}_t))/\sqrt{\bar{\alpha}_{t-1}}$, and in light of Assumption 2 and 3, it is known that $p_\theta(\mathbf{x}_t\,|\mathcal{D}_t(\sqrt{\bar{\alpha}_t}\,\mathbf{h}_t))/\sqrt{\bar{\alpha}_{t-1}} \sim \mathcal{N}(\mathbf{x}, \frac{1-\bar{\alpha}_{t-1}}{\bar{\alpha}_{t-1}}\,\mathbf{I})$. Hence, according to Eq. (5), we have $\tilde{\sigma}_{t-1}^2 = \frac{1-\bar{\alpha}_{t-1}}{\bar{\alpha}_{t-1}}$. Let $\mathbf{x}_t := \sqrt{\bar{\alpha}_t}\tilde{\mathbf{x}}_t$, $\mathbf{s}_t := \sqrt{\bar{\alpha}_t}\tilde{\mathbf{s}}_t$, and substitute into Eq. (4), we obtain:

$$\begin{aligned}
\mathbf{s}_t &= \mathbf{x}_t - \sqrt{\bar{\alpha}_t}\,\mathbf{\Phi}^{\mathrm{T}}(\mathbf{\Phi}\,\tilde{\mathbf{x}}_t - \mathbf{y}) \\
\sqrt{\bar{\alpha}_t}\,\mathbf{h}_t &= \mathbf{s}_t + \sqrt{\bar{\alpha}_t}o_t(\mathbf{u}_{t+1}, \mathbf{h}_{t+1}) \\
\mathbf{x}_{t-1} &= p_\theta(\mathbf{x}_{t-1}\,|\mathcal{D}_t(\sqrt{\bar{\alpha}_t}\,\mathbf{h}_t))
\end{aligned} \tag{6}$$

and the state evolution can be rewrite as follows:

$$\begin{aligned}
\sqrt{\bar{\alpha}_t}\,\mathbf{h}_t &\sim \mathcal{N}(\sqrt{\bar{\alpha}_t}\,\mathbf{x}, \bar{\alpha}_t\,\tilde{\sigma}_t^2\,\mathbf{I}/\delta) \\
\tilde{\sigma}_{t-1}^2 &= \mathbb{E}\left\{[p_\theta(\mathbf{x}_{t-1}\,|\mathcal{D}_t(\sqrt{\bar{\alpha}_t}\,\mathbf{h}_t))/\sqrt{\bar{\alpha}_{t-1}} - \mathbf{x}]^2\right\} \\
&= \frac{1}{\bar{\alpha}_{t-1}}\mathbb{E}\left\{[p_\theta(\mathbf{x}_{t-1}\,|\mathcal{D}_t(\sqrt{\bar{\alpha}_t}\,\mathbf{h}_t)) - \sqrt{\bar{\alpha}_{t-1}}\,\mathbf{x}]^2\right\}
\end{aligned} \tag{7}$$

define $\sigma_t := \sqrt{\bar{\alpha}_t}\tilde{\sigma}_t$, we can get the following equation:

$$\begin{aligned}
\sqrt{\bar{\alpha}_t}\,\mathbf{h}_t &\sim \mathcal{N}(\sqrt{\bar{\alpha}_t}\,\mathbf{x}, \sigma_t^2\,\mathbf{I}/\delta) \\
\sigma_{t-1}^2 &= \mathbb{E}\left\{[p_\theta(\mathbf{x}_{t-1}\,|\mathcal{D}_t(\sqrt{\bar{\alpha}_t}\,\mathbf{h}_t)) - \sqrt{\bar{\alpha}_{t-1}}\,\mathbf{x}]^2\right\}
\end{aligned} \tag{8}$$

Finally, define $\mathbf{r}_t := \mathcal{D}_t(\sqrt{\bar{\alpha}_t}\mathbf{h}_t)$, and substitute into Eq. (6) and (8) to obtain the following Theorem:

Theorem 1. *Suppose Assumption 1, 2, and 3 hold, we deduce the iterative representation of diffusion message passing (DMP) algorithm as follows:*

$$\begin{aligned}
\mathbf{s}_t &= \mathbf{x}_t - \sqrt{\bar{\alpha}_t}\,\mathbf{\Phi}^{\mathrm{T}}(\mathbf{\Phi}\mathbf{x}_t - \mathbf{y}) \\
\mathbf{r}_t &= \mathcal{D}_t[\mathbf{s}_t + \sqrt{\bar{\alpha}_t}o_t(\mathbf{u}_{t+1}, \mathbf{h}_{t+1})] \\
\mathbf{x}_{t-1} &= p_\theta(\mathbf{x}_{t-1}\,|\,\mathbf{r}_t)
\end{aligned} \tag{9}$$

where $o_t(\mathbf{u}_{t+1}, \mathbf{h}_{t+1}) := \mathbf{\Phi}^{\mathrm{T}}\,\mathbf{u}_{t+1}\,\mathrm{div}\eta_t(\mathbf{h}_{t+1})$ *.Moreover, it is possible to write out its state evolution:*

$$\begin{aligned}
\mathbf{r}_t &\sim \mathcal{N}(\sqrt{\bar{\alpha}_t}\,\mathbf{x}, \sigma_t^2\,\mathbf{I}) \\
\sigma_{t-1}^2 &= \mathbb{E}\left\{[p_\theta(\mathbf{x}_{t-1}\,|\,\mathbf{r}_t) - \sqrt{\bar{\alpha}_{t-1}}\,\mathbf{x}]^2\right\}
\end{aligned} \tag{10}$$