Improving Semi-Supervised Semantic Segmentation with Sliced-Wasserstein Feature Alignment and Uniformity

Supplementary Material

Proof: Uniform Distribution of Normalized Gaussian Vectors

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be an n-dimensional random vector where each $X_i \overset{iid}{\sim} N(0,1)$. Then the normalized vector

$$\mathbf{Y} = \frac{\mathbf{X}}{\|\mathbf{X}\|}$$

is uniformly distributed over the unit hypersphere S^{n-1} in $\mathbb{R}^n$.

Proof. We proceed in several steps to show that $\mathbf{Y}$ inherits a uniform distribution from the spherical symmetry of the multivariate normal distribution.

Joint Distribution of $\mathbf{X}$ Since the components of $\mathbf{X}$ are independent standard normal variables, their joint probability density function is:

$$f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-x_i^2/2} = \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{\|\mathbf{x}\|^2}{2}\right)$$

where $\|\mathbf{x}\|^2 = \sum_{i=1}^{n} x_i^2$. Crucially, this density depends only on $\|\mathbf{x}\|$, indicating spherical symmetry.

Spherical Coordinate Transformation We transform to n-dimensional spherical coordinates. The volume element in these coordinates is:

$$dV = r^{n-1}\, dr\, d\Omega$$

where:

- $r = \|\mathbf{x}\|$ is the radial coordinate
- $d\Omega$ is the surface element on the unit sphere S^{n-1}

Distribution in Spherical Coordinates In spherical coordinates, the joint density becomes:

$$f_{\mathbf{X}}(r, \Omega) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{r^2}{2}\right)$$

This expression can be factored as:

$$f_{\mathbf{X}}(r, \Omega) = f_r(r) \cdot f_\Omega(\Omega)$$

where:

$$f_r(r) = C_n \cdot r^{n-1} \exp\left(-\frac{r^2}{2}\right)$$

and $f_\Omega(\Omega)$ is constant over the sphere. Here, C_n is a normalizing constant.

Independence and Uniformity Two key observations follow:

- The radial component r and the angular components Ω are independent, as evidenced by the factorization.
- The angular density $f_\Omega(\Omega)$ is constant, implying a uniform distribution over S^{n-1}.

Distribution of $\mathbf{Y}$ The normalized vector $\mathbf{Y} = \mathbf{X}/\|\mathbf{X}\|$ depends only on the angular components:

$$\mathbf{Y} = (\cos\theta_1, \sin\theta_1 \cos\theta_2, \ldots, \prod_{k=1}^{n-2} \sin\theta_k \cos\theta_{n-1}, \prod_{k=1}^{n-1} \sin\theta_k)$$

Since the angular components are uniformly distributed on S^{n-1}, so is $\mathbf{Y}$. □