UniPhy: Learning a Unified Constitutive Model for Inverse Physics Simulation

Supplementary Material

1. Implementation Details

Teacher Forcing: The baseline, NCLaw [4] uses a teacher-forcing scheme that restarts the predicted simulation from ground truth state periodically. The ground truth state includes the position $\mathbf{x}$, velocity $\mathbf{v}$, affine velocity $\mathbf{C}$, and deformation gradient $\mathbf{F}$. This period starts from 25 steps and is increased to 200 by a cosine annealing scheduler. This introduces the privileged information of position, velocity, affine velocity, and deformation gradient from the simulator during inference. Since the privileged information and access to the simulator may not be available during inference, we evaluate our method with the baseline NCLaw on the setting of without teacher-forcing and report the results in Table 1 and Table 2 of the main paper and show the visualizations in the webpage attached in the supplementary.

2. Algorithm

We detail our training and inference algorithm in Algorithm 1 and Algorithm 2 respectively where i is the trajectory index, p represents the particle index and t represents the time.

3. Analytical Constitutive Laws

In this section, we discuss the constitutive model and the deformation gradient projection/mapping function for the materials that are used to simulate the trajectories used in training.

In Material Point Method (MPM), each particle has a deformation gradient F which is projected on to the yield surface using a return mapping $\mathcal{G}$. This projected deformation gradient is then used by the constitutive law to compute the internal forces experienced by the particle given as the Cauchy stress $\mathcal{S}$.

Elastic: As there is no plasticity in elastic materials, the deformation gradient projection is an identity function defined as:

$$\mathcal{G}(F) = F \tag{1}$$

We use the neo-Hookean elasticity model for elastic materials. The Cauchy stress for the elastic material is calculated as:

$$J\mathcal{S}(\mathbf{F}) = \mu(\mathbf{F}\mathbf{F}^\top) + (\lambda \log(J) - \mu)\mathbf{I} \tag{2}$$

where μ and λ are the Lamé parameters of the Young's modulus and Poisson ratio. The Young's modulus defines the stiffness of the material and Poisson ratio defines the ability of the object to preserve its volume under deformation.

For elastic materials, we have a range of [350.0, 2595196.0] for μ and a range of [500.0, 2580120.0] for λ.

Newtonian: The stress for the newtonian fluid is computed as:

$$\kappa = \frac{2}{3}\mu + \lambda \tag{3}$$

$$J\mathcal{S}(\mathbf{F}) = \kappa\mathbf{I}(J - \frac{1}{J^6}) + \frac{1}{2}\mu\left(\nabla\mathbf{v} + \nabla\mathbf{v}^\top\right) \tag{4}$$

where $\nabla\mathbf{v}$ is the affine velocity of the particle $\mathbf{C}$, μ represents the velocity change opposition and κ is volume preservation ability.

For newtonian materials, we have a range of [50.0, 1e3] for μ and a range of [30.0, 5e5] for λ.

Plasticine: For plasticine materials, we use the von-Mises plastic return mapping for deformation gradient. The SVD of $\mathbf{F}$ can be defined as $F = U\Sigma V$ where $\epsilon = log(\Sigma)$ is the Hencky strain. The von Mises yield condition is defined as:

$$\delta\gamma = \|\hat{\boldsymbol{\epsilon}}\| - \frac{\tau_Y}{2\mu} \tag{5}$$

where $\boldsymbol{\epsilon}$ is the normalized Hencky strain, τ_Y is the yield stress determining the plastic flow and the stress required for causing permanent deformation/yielding behavior. In the above yielding condition, if $\delta\gamma > 0$, then the deformation gradient breaks the yield constraint and is projected back into the elastic region via the following mapping:

$$\mathcal{G}(\mathbf{F}) = \begin{cases} \mathbf{F} & \delta\gamma \leq 0 \\ \mathbf{U}\exp\left(\boldsymbol{\epsilon} - \delta\gamma\frac{\hat{\boldsymbol{\epsilon}}}{\|\hat{\boldsymbol{\epsilon}}\|}\right)\mathbf{V}^\top & \delta\gamma > 0 \end{cases}$$

To calculate stress, we use the St.Venant-Kirchhoff (StVK) constitutive model.

$$J\mathbf{S}(\mathbf{F}) = \mathbf{U}(2\mu\boldsymbol{\epsilon} + \lambda\,\mathrm{Tr}(\boldsymbol{\epsilon}))\mathbf{U}^\top \tag{6}$$

For plasticine materials, we have a range of [1e4, 1e6] for μ, a range of [1e4, 3e6] for λ and a range of [5e3, 1e4] for τ_Y.

Sand: To simulate sand particles, we use the Drucker-Prager [3] yield criteria as follows:

$$\mathrm{tr}(\boldsymbol{\epsilon}) > 0, \quad \text{or} \quad \delta\gamma = \|\hat{\boldsymbol{\epsilon}}\|_F + \alpha\frac{(3\lambda + 2\mu)\,\mathrm{tr}(\boldsymbol{\epsilon})}{2\mu} > 0. \tag{7}$$

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Algorithm 1 UniPhy: Training

Input: Dataset of trajectories $\mathcal{D} = \{\mathbf{F}, \mathbf{F}_{proj}, \mathbf{C}, \mathbf{S}\}$
Output: ϕ, θ, z
for $iteration = 1, 2, \ldots,$ N **do**
  Batch of n samples, $\mathcal{B} = (\mathbf{F}_i^{p,t}, \mathbf{F}_{proj,i}^{p,t}, \mathbf{S}_i^{p,t}, \mathbf{C}_i^{p,t}, z_i)$
  $\mathbf{F}_i^{p,t} \overset{\text{SVD}}{=} \boldsymbol{U\Sigma V}^T$
  $\Delta = g_\phi(\mathbf{F}_i^{p,t}, \boldsymbol{U}, \boldsymbol{V}^T, z_i)$
  $\hat{\mathbf{F}}_i^{p,t} = \mathbf{F}_i^{p,t} + \Delta$
  $\mathbf{F} = \mathbf{F}_{proj,i}^{p,t}$
  $\mathbf{F}_{max} = max(\mathbf{F}[:, 0, 0], 1e-6)$
  $\mathbf{F} \overset{\text{SVD}}{=} \boldsymbol{U}_{proj}\boldsymbol{\Sigma}_{proj}\boldsymbol{V}_{proj}$
  $\boldsymbol{R} = \boldsymbol{U}_{proj}\boldsymbol{V}_{proj}^T$
  $\boldsymbol{S}_1 = f_\theta(\boldsymbol{\Sigma}, \mathbf{F}^T\mathbf{F}, \det(\boldsymbol{F}), log(\det(\boldsymbol{F})), \mathbf{F}_{max}, log(\mathbf{F}_{max}), \mathbf{C}_i^{p,t}, z_i)$
  $\hat{\mathbf{S}}_i^{p,t} = \frac{1}{2}\left(\boldsymbol{S}_1 + \boldsymbol{S}_1^T\right)$
  $\underset{\theta,\phi,z}{min}\left(\mathcal{L}(\hat{\mathbf{F}}_i^{p,t}, \mathbf{F}_{proj,i}^{p,t}) + \mathcal{L}(\hat{\mathbf{S}}_i^{p,t}, \mathbf{S}_i^{p,t}) + \frac{1}{\sigma^2}\|z_i\|^2\right)$
  Optimize ϕ, θ, z
end for

Algorithm 2 UniPhy: Inference using Differentiable Material Point Method (MPM)

Input: $\mathbf{x}, z, f_\theta, g_\phi$
Output: $\hat{z}$
for $epoch = 1, 2, \ldots,$N **do**
  for $iteration = 1, 2, \ldots,$t **do**
    Transfer mass and momentum of particles to grid nodes
    $\mathbf{F}^{p,t+1} = (I + \Delta t * \mathbf{C}^{p,t}) * \mathbf{F}^{p,t}$
    $\mathbf{F}^{p,t+1} \overset{\text{SVD}}{=} \boldsymbol{U\Sigma V}^T$
    $\Delta = g_\phi(\mathbf{F}^{p,t+1}, \mathbf{U}, \mathbf{V}^T, z_i)$
    $\mathbf{F}_{proj}^{p,t+1} = \mathbf{F}^{p,t+1} + \Delta$
    $\mathbf{F}_{proj}^{p,t+1} \overset{\text{SVD}}{=} \mathbf{U}_{proj}\boldsymbol{\Sigma}_{proj}\mathbf{V}_{proj}$
    $\boldsymbol{R}_{proj} = \mathbf{U}_{proj}\mathbf{V}_{proj}^T$
    $\boldsymbol{S}_1 = f_\theta(\boldsymbol{\Sigma}, \mathbf{F}^T\mathbf{F}, \det(\boldsymbol{F}), log(\det(\boldsymbol{F})), \mathbf{F}_{max}, log(\mathbf{F}_{max}), \mathbf{C}_i^{p,t}, z_i)$
    $\hat{\mathbf{S}}_i^{p,t} = \frac{1}{2}\left(\boldsymbol{S}_1 + \boldsymbol{S}_1^T\right)$
    Update momentum and velocity of grid node
    Transfer momentum and velocity from grid node to particle
    Advect particles $\hat{\mathbf{x}}^{t+1} = \mathbf{x}^t + \Delta t \mathbf{v}^{t+1}$
  end for
  $\hat{z} = \underset{z}{min}\,\mathcal{L}(\hat{x}, x)$
  Optimize $\hat{z}$
end for

where $\alpha = \sqrt{\frac{2}{3}}\frac{2\sin\theta_{\text{fric}}}{3-\sin\theta_{\text{fric}}}$ and θ_{fric} is the friction angle determining the slope of the sand pile. Then, we use the deformation gradient projection function as follows:

$$\mathcal{G}(\mathbf{F}) = \begin{cases} \mathbf{U}\mathbf{V}^\top & \text{tr}(\boldsymbol{\epsilon}) > 0 \\ \mathbf{F} & \delta\gamma \leq 0 \& \, \text{tr}(\boldsymbol{\epsilon}) \leq 0 \\ \mathbf{U}\exp\left(\boldsymbol{\epsilon} - \delta\gamma\frac{\hat{\epsilon}}{\|\epsilon\|}\right)\mathbf{V}^\top & \delta\gamma > 0 \& \, \text{tr}(\boldsymbol{\epsilon}) \leq 0 \end{cases} \tag{8}$$

For sand materials, we have a range of [2400.0, 9e6] for μ, a range of [2400.0, 9e6] for λ and a range of [0.01, 0.4] for θ_{fric} .

Non-newtonian: To model non-newtonian materials [1], we use the viscoplastic model [2, 5] and von-Mises criteria to define the elastic region. Having the viscoplastic model prevents the deformation from being directly mapped back onto the yield surface. Non-newtonian materials have yield stress as well. We define the deformation gradient return mapping as follows:

$$\hat{\mu} = \frac{\mu}{d} \operatorname{Tr}\left(\boldsymbol{\Sigma}^2\right) \tag{9}$$

$$s = 2\mu\hat{\epsilon} \tag{10}$$

$$\hat{s} = \|\boldsymbol{s}\| - \frac{\delta\gamma}{1 + \frac{\eta}{2\hat{\mu}\Delta t}} \tag{11}$$

$$\mathcal{Z}(\mathbf{F}) = \begin{cases} \mathbf{F} & \delta\gamma \leq 0 \\ \mathbf{U} \exp\left(\frac{\hat{s}}{2\mu}\hat{\boldsymbol{\epsilon}} + \frac{1}{d}\operatorname{Tr}(\boldsymbol{\epsilon})\mathbf{1}\right)\mathbf{V}^\top & \delta\gamma > 0 \end{cases} \tag{12}$$

For non-newtonian materials, we have a range of [1e3, 2e6] for μ, a range of [1e3, 2e6] for λ, a range of [1e3, 2e6] for τ_Y and a range of [0.1, 100.0] for η.

References

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[3] Gergely Klár, Theodore Gast, Andre Pradhana, Chuyuan Fu, Craig Schroeder, Chenfanfu Jiang, and Joseph Teran. Drucker-prager elastoplasticity for sand animation. *ACM Transactions on Graphics (TOG)*, 35(4), 2016. 1

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