A1. Appendix

A1.1. Theoretical Details

Clarification on Parameter Settings To eliminate any potential confusion, we formally clarify the parameter settings:

1. The essential condition:

$$\boldsymbol{V_{\mathrm{thr}} = \frac{q_{\mathrm{thr}}}{C}}$$

   must be strictly satisfied for our theorems to hold.

2. The default value of $\mathbf{V_0}$ is:

$$\boldsymbol{V_0 = (0.5q_{\mathrm{thr}} + \mathrm{bias})/C},$$

   where bias is the bias term in a convolutional or linear layer.

3. Unless explicitly stated otherwise, the default values for the remaining parameters are:

$$\boldsymbol{S_0 = 0}, \quad \boldsymbol{S_{\min} = 0}.$$

A1.1.1. Preliminaries

In this section, we describe the neuron dynamics and the feed-forward network model used in our analysis. We introduce the notations and mathematical formulations that underpin the behavior of neurons within the network.

Neuron Dynamics. Each neuron in the network follows the ST-BIF (Bipolar Integrate & Fire with Spike Tracer) model. The neuron's behavior is defined by the following equations:

At each time step t, the neuron's membrane potential $\hat{V}_t$ is updated based on the previous potential V_{t-1} and the input charges received from presynaptic neurons:

$$\hat{V}_t = V_{t-1} + (\sum_{i=1}^{m} w_i \cdot q_{i,t})/C \tag{A1}$$

Here C is the membrane capacitance, w_i are the synaptic weights, and $q_{i,t}$ are the input charges at time t from m presynaptic neurons.

The neuron decides whether to emit a spike based on the updated membrane potential $\hat{V}_t$, the threshold voltage V_{thr}, and the previous spike tracer value S_{t-1}:

$$\Theta(\hat{V}_t, V_{\mathrm{thr}}, S_{t-1}) = \begin{cases} 1, & \text{if } \hat{V}_t \geq V_{\mathrm{thr}} \text{ and } S_{t-1} < S_{\max} \\ 0, & \text{if } 0 \leq \hat{V}_t < V_{\mathrm{thr}} \\ -1, & \text{if } \hat{V}_t < 0 \text{ and } S_{t-1} > S_{\min} \end{cases} \tag{A2}$$

where $S_{\max}$ and $S_{\min}$ are the upper and lower limits for the spike tracer S_t.

The output charge of the neuron at time t depends on the spike decision:

$$q_t^{\mathrm{out}} = q_{\mathrm{thr}} \cdot \Theta(\hat{V}_t, V_{\mathrm{thr}}, S_{t-1}), \tag{A3}$$

where q_{thr} is the threshold charge associated with a spike.

After determining the output, the membrane potential is adjusted to account for the emitted charge, and the spike tracer S_t is updated to record the neuron's firing history:

$$V_t = \hat{V}_t - q_t^{\mathrm{out}}/C; \quad S_t = S_{t-1} + \Theta(\hat{V}_t, V_{\mathrm{thr}}, S_{t-1}) \tag{A4}$$

Feed-forward Network Model. The network is structured as a feed-forward architecture with L layers, indexed from 0 (input layer) to L_0 (output layer). The neurons in each layer receive inputs from the previous layer and send outputs to the next layer. There are no recurrent connections.

We use supersript (l) and subscript i to denote a quantity of the i-th neuron at layer l. So the temporary potential update of the i-th neuron at layer l can be expressed as:

$$\hat{V}_{i,t}^{(l)} = V_{i,t-1}^{(l)} + (\sum_{j=1}^{m_{l-1}} q_{j,t}^{(l-1),\mathrm{out}} \cdot w_{j,i}^{(l)})/C \tag{A5}$$

Notations. Here is a summary of mathematical notations used in the proof:

- **Threshold charge**: q_{thr} is the threshold charge for spiking.
- **Membrane capacitance**: C is the membrane capacitance.
- **Input charge**: $q_{j_0,t}$ is the input charge at time t for input neuron j_0 (in layer 0).
- **Layers** (l): The network consists of multiple layers, with neurons in layer l receiving inputs from neurons in layer $l-1$.
- **Synaptic weight**: $w_{j_k,j_{k+1}}^{(k+1)}$ is the synaptic weight from neuron j_k in layer k to neuron j_{k+1} in layer $k+1$.
- **Membraine potential**: V_T is the potential of neuron membrane at time t.
- **Total input charge**: $Q_{j_0,T} = \sum_{t=1}^{T} q_{j_0,t}^{\mathrm{in}}$ is the total input charge by the input neuron j_0.
- **Input charges**: $q_t^{\mathrm{in}} = \sum_{i=1}^{m} w_i \, q_{i,t}$ is the input charges received at time t.
- **Total input charge for a neuron**: $Q_T^{\mathrm{in}} = \sum_{t=1}^{T} q_t^{\mathrm{in}}$ is the total input charge from presynaptic neurons up to time T. Note that w is timed here and the quantity should be distinguished from $Q_{j_0,T}$ by the superscript $(\cdot)^{\mathrm{in}}$.
- **Spike tracer**: S_t is the spike tracer value at time t.
- **Equilibrium Time**: T_{eq} is the time-step that neuron enters equilibrium state.
- **Clip**: $\mathrm{clip}(x, \alpha_{\min}, \alpha_{\max})$ is an operator that limits x between $\alpha_{\min}$ and $\alpha_{\max}$.

- **Round**: round(x) rounds the input x to the nearest integer.
- **Floor**: $\mathrm{floor}(x) = \lfloor x \rfloor$ rounds the input x down to the greatest integer less than or equal to x.
- **QReLU**: QReLU is a function that quantized $\mathrm{ReLU}(x)$ and can take only m values at all, the precise expression is $\mathrm{QReLU}(x) = \alpha\,\mathrm{clip}(\mathrm{round}(x/\alpha), 0, m-1)$. Here α and m are two parameters with the function which can be adjusted. m is an integer. (Note that we have $\mathrm{QReLU}(x) = \alpha\,\mathrm{clip}(\lfloor (x/\alpha) + 0.5 \rfloor, 0, m-1)$.)
- **ANN with Q-ReLU activation functions**: $\mathbf{F}_{\mathrm{QReLU}}(\mathbf{x};\theta)$ represents the output of an ANN with Q-ReLU activation functions, with input x and parameters θ. In particular, we use $\mathbf{F}^{(l)}_{\mathrm{QReLU}}$ to represent an artificial neural network (ANN) with l layers. The parameters of the l-th layer are denoted by $\theta^{(l)}$, and the parameters spanning layers l_1 to l_2 are represented by $\theta^{[l_1:l_2]}$.

A1.1.2. Theoretical Details for a Single Neuron

Lemma A1.1 (Spike Tracer and Output Charge) *For an ST-BIF neuron over time steps from $t = 0$ to $t = T$, the total output charge is proportional to the change in the spike tracer:*

$$\sum_{t=1}^{T} q_t^{\mathrm{out}} = q_{\mathrm{thr}}(S_T - S_0)$$

Proof. From the spike tracer update equation:

$$S_t = S_{t-1} + \Theta(\hat{V}_t, V_{\mathrm{thr}}, S_{t-1})$$

Since:

$$q_t^{\mathrm{out}} = \Theta(\hat{V}_t, V_{\mathrm{thr}}, S_{t-1}) \times q_{\mathrm{thr}}$$

We can write:

$$\Theta(\hat{V}_t, V_{\mathrm{thr}}, S_{t-1}) = \frac{q_t^{\mathrm{out}}}{q_{\mathrm{thr}}}$$

Substituting back into the spike tracer update:

$$S_t = S_{t-1} + \frac{q_t^{\mathrm{out}}}{q_{\mathrm{thr}}}$$

Summing both sides from $t = 1$ to T:

$$\sum_{t=1}^{T} S_t = \sum_{t=1}^{T} \left(S_{t-1} + \frac{q_t^{\mathrm{out}}}{q_{\mathrm{thr}}} \right)$$

Observing that $\sum_{t=1}^{T} S_{t-1} = \sum_{t=0}^{T-1} S_t$, the equation simplifies to:

$$S_T - S_0 = \sum_{t=1}^{T} \frac{q_t^{\mathrm{out}}}{q_{\mathrm{thr}}}$$

Multiplying both sides by q_{thr}:

$$q_{\mathrm{thr}}(S_T - S_0) = \sum_{t=1}^{T} q_t^{\mathrm{out}}$$

Which proves the lemma. □

Theorem A1.2 (LoCC for a Single Neuron) *For a single ST-BIF neuron, the neuron obeys the Law of Charge Conservation at any time step T*

$$C(V_T - V_0) + q_{\mathrm{thr}}(S_T - S_0) - Q_T^{\mathrm{in}} = 0, \tag{A6}$$

Proof. We start with the membrane potential update equation for the ST-BIF neuron:

$$V_t = V_{t-1} + \frac{q_t^{\mathrm{in}}}{C} - \frac{q_t^{\mathrm{out}}}{C} \tag{A7}$$

Subtract V_{t-1} from both sides to find the change in membrane potential:

$$V_t - V_{t-1} = \frac{q_t^{\mathrm{in}}}{C} - \frac{q_t^{\mathrm{out}}}{C} \tag{A8}$$

Summing both sides from $t = 1$ to $t = T$, we get:

$$\sum_{t=1}^{T} (V_t - V_{t-1}) = \frac{1}{C} \sum_{t=1}^{T} \left(q_t^{\mathrm{in}} - q_t^{\mathrm{out}} \right) \tag{A9}$$

The left-hand side simplifies to:

$$V_T - V_0 = \frac{1}{C} \left(\sum_{t=1}^{T} q_t^{\mathrm{in}} - \sum_{t=1}^{T} q_t^{\mathrm{out}} \right) \tag{A10}$$

Let us denote:

$$Q_T^{\mathrm{in}} = \sum_{t=1}^{T} q_t^{\mathrm{in}} \tag{A11}$$

$$Q_T^{\mathrm{out}} = \sum_{t=1}^{T} q_t^{\mathrm{out}} \tag{A12}$$

Thus, we have:

$$V_T - V_0 = \frac{1}{C} \left(Q_T^{\mathrm{in}} - Q_T^{\mathrm{out}} \right) \tag{A13}$$

From Lemma A1.1, we know that:

$$Q_T^{\mathrm{out}} = q_{\mathrm{thr}}(S_T - S_0) \tag{A14}$$

Substituting this back into our equation:

$$V_T - V_0 = \frac{1}{C} \left(Q_T^{\mathrm{in}} - q_{\mathrm{thr}}(S_T - S_0) \right) \tag{A15}$$

Multiplying both sides by C, we obtain:

$$C(V_T - V_0) = Q_T^{\mathrm{in}} - q_{\mathrm{thr}}(S_T - S_0) \tag{A16}$$

Rearranging terms, we arrive at the Law of Charge Conservation for a single ST-BIF neuron:

$$C(V_T - V_0) + q_{\text{thr}}(S_T - S_0) - Q_T^{\text{in}} = 0 \tag{A17}$$

This completes the proof. □

Lemma A1.3 (Equilibium State Lemma) *After entering the equilibrium state at time T_{eq}, the membrane potential $V_{T_{\text{eq}}}$ and the spike tracer $S_{T_{\text{eq}}}$ of an ST-BIF neuron can only be in one of the following three states:*
1. $0 \le V_{T_{\text{eq}}} < V_{\text{thr}}, \quad S_{\min} \le S_{T_{\text{eq}}} \le S_{\max}$,
2. $V_{T_{\text{eq}}} \ge V_{\text{thr}}, \quad S_{T_{\text{eq}}} = S_{\max}$,
3. $V_{T_{\text{eq}}} < 0, \quad S_{T_{\text{eq}}} = S_{\min}$.

Proof. This follows directly from the dynamics of the ST-BIF neuron. In the absence of additional input, the neuron reaches equilibrium state when:
1. The membrane potential is subthreshold and stable: $0 \le V_t < V_{\text{thr}}$.
2. The membrane potential has reached or exceeded the threshold, but the spike tracer is at its maximum value: $V_t \ge V_{\text{thr}}$, $S_t = S_{\max}$. In this case, the neuron cannot produce additional spikes due to the spike tracer limit.
3. The membrane potential is negative, and the spike tracer is at its minimum value: $V_t < 0$, $S_t = S_{\min}$. The neuron cannot decrease its spike count further.

These conditions define the possible equilibrium states of the neuron after input cessation. □

Theorem A1.4 (Temporal Independence at Equilibrium) *For an ST-BIF neuron with no input after time step T, the spike tracer $S_{T_{\text{eq}}}$ at equilibrium (no more spikes after T_{eq}) is determined as*

$$S_{T_{\text{eq}}} = \text{clip}\left(S_0 + \left\lfloor \frac{1}{q_{\text{thr}}}\left(CV_0 + Q_T^{\text{in}}\right)\right\rfloor,\ S_{\min},\ S_{\max}\right).$$

Considering the default values $S_0 = 0,\ V_0 = (0.5q_{\text{thr}} + \text{bias})/C,\ S_{\min} = 0$, the output charge satisfies $q_{\text{thr}} S_{T_{\text{eq}}} = \text{QReLU}(Q_T^{\text{in}} + \text{bias})$.

Proof. From the Law of Charge Conservation (LoCC) for a single neuron, we have:

$$C\left(V_{T_{\text{eq}}} - V_0\right) + q_{\text{thr}}\left(S_{T_{\text{eq}}} - S_0\right) - Q_{\text{total}}^{\text{in}} = 0. \tag{A18}$$

Rewriting equation A18, we obtain:

$$V_{T_{\text{eq}}} = V_0 + \frac{Q_{\text{total}}^{\text{in}}}{C} - \frac{q_{\text{thr}}}{C}\left(S_{T_{\text{eq}}} - S_0\right). \tag{A19}$$

We proceed to analyze the possible cases for $V_{T_{\text{eq}}}$ and $S_{T_{\text{eq}}}$ based on the Equilibrium State Lemma.

Case 1: $0 \le V_{T_{\text{eq}}} < V_{\text{thr}}, \quad S_{\min} \le S_{T_{\text{eq}}} \le S_{\max}$.
From equation A19, substituting $V_{\text{thr}} = \frac{q_{\text{thr}}}{C}$:

$$0 \le V_0 + \frac{Q_{\text{total}}^{\text{in}}}{C} - \frac{q_{\text{thr}}}{C}\left(S_{T_{\text{eq}}} - S_0\right) < \frac{q_{\text{thr}}}{C}. \tag{A20}$$

Rearranging the equation, we have:

$$S_{T_{\text{eq}}} - S_0 \le \frac{CV_0 + Q_{\text{total}}^{\text{in}}}{q_{\text{thr}}} < S_{T_{\text{eq}}} - S_0 + 1. \tag{A21}$$

$S_{T_{\text{eq}}} - S_0$ should be an integer, so we have

$$S_{T_{\text{eq}}} = S_0 + \left\lfloor \frac{CV_0 + Q_{\text{total}}^{\text{in}}}{q_{\text{thr}}} \right\rfloor. \tag{A22}$$

Applying the constraints $S_{\min} \le S_{T_{\text{eq}}} \le S_{\max}$, we have:

$$S_{T_{\text{eq}}} = \text{clip}\left(S_0 + \left\lfloor \frac{CV_0 + Q_{\text{total}}^{\text{in}}}{q_{\text{thr}}} \right\rfloor,\ S_{\min},\ S_{\max}\right). \tag{A23}$$

Case 2: $V_{T_{\text{eq}}} \ge V_{\text{thr}}, \quad S_{T_{\text{eq}}} = S_{\max}$.
From Equation equation A19:

$$V_{T_{\text{eq}}} = V_0 + \frac{Q_{\text{total}}^{\text{in}}}{C} - \frac{q_{\text{thr}}}{C}\left(S_{\max} - S_0\right). \tag{A24}$$

Since $V_{T_{\text{eq}}} \ge V_{\text{thr}}$, we have:

$$V_0 + \frac{Q_{\text{total}}^{\text{in}}}{C} - \frac{q_{\text{thr}}}{C}\left(S_{\max} - S_0\right) \ge V_{\text{thr}} = \frac{q_{\text{thr}}}{C}. \tag{A25}$$

Rearranging the equation, we have:

$$CV_0 + Q_{\text{total}}^{\text{in}} \ge q_{\text{thr}}\left(S_{\max} - S_0 + 1\right). \tag{A26}$$

This implies:

$$S_0 + \left\lfloor \frac{CV_0 + Q_{\text{total}}^{\text{in}}}{q_{\text{thr}}} \right\rfloor \ge S_{\max} + 1. \tag{A27}$$

Therefore, after applying the clipping function:

$$S_{T_{\text{eq}}} = \text{clip}\left(S_{\max} + 1,\ S_{\min},\ S_{\max}\right) = S_{\max}. \tag{A28}$$

Case 3: $V_{T_{\text{eq}}} < 0, \quad S_{T_{\text{eq}}} = S_{\min}$.
From Equation equation A19:

$$V_{T_{\text{eq}}} = V_0 + \frac{Q_{\text{total}}^{\text{in}}}{C} - \frac{q_{\text{thr}}}{C}\left(S_{\min} - S_0\right). \tag{A29}$$

Since $V_{T_{\text{eq}}} < 0$, we have:

$$V_0 + \frac{Q^{\text{in}}_{\text{total}}}{C} - \frac{q_{\text{thr}}}{C}\left(S_{\min} - S_0\right) < 0. \tag{A30}$$

Rearranging the equation, we have:

$$CV_0 + Q^{\text{in}}_{\text{total}} < q_{\text{thr}}\left(S_{\min} - S_0\right). \tag{A31}$$

This implies:

$$S_0 + \left\lfloor \frac{CV_0 + Q^{\text{in}}_{\text{total}}}{q_{\text{thr}}} \right\rfloor < S_{\min}. \tag{A32}$$

Therefore, after applying the clipping function:

$$S_{T_{\text{eq}}} = \text{clip}\left(S_{\min} - 1,\ S_{\min},\ S_{\max}\right) = S_{\min}. \tag{A33}$$

Conclusion:

In all cases, the spike tracer at equilibrium can be expressed as:

$$S_{T_{\text{eq}}} = \text{clip}\left(S_0 + \left\lfloor \frac{CV_0 + Q^{\text{in}}_{\text{total}}}{q_{\text{thr}}} \right\rfloor,\ S_{\min},\ S_{\max}\right). \tag{A34}$$

Substituting this back into Equation equation A19, we find that $V_{T_{\text{eq}}}$ is fully determined by $Q^{\text{in}}_{\text{total}}$, independent of the timing of the inputs.

This demonstrates that the total output spike count (encoded in $S_{T_{\text{eq}}}$) depends only on the total input charge $Q^{\text{in}}_{\text{total}}$, and not on how the inputs are distributed over time. Moreover, with the default values $S_0 = 0,\ V_0 = (0.5q_{\text{thr}} + \text{bias})/C,\ S_{\min} = 0$, the total outputs is

$$\begin{aligned} q_{\text{thr}} S_{T_{\text{eq}}} =& q_{\text{thr}}\ \text{clip}\left(\left\lfloor \frac{1}{q_{\text{thr}}}(Q^{\text{in}}_T + \text{bias}) + 0.5 \right\rfloor, 0,\ S_{\max}\right) \\ =& \text{QReLU}(Q^{\text{in}}_T + \text{bias}), \end{aligned}$$

Here the two operators α and m for QReLU take value q_{thr} and $S_{\max} + 1$ respectively.

□

A1.1.3. Theoretical Details for a Neural Network

Theorem A1.5 (LoCC for a Network) *Consider a feed-forward ST-BIF neural network with L_0 layers. For any neuron i in layer L ($1 \le L \le L_0$), the Law of Charge Conservation is given by*

$$\begin{aligned} &C\left(V^{(L)}_{i,T} - V^{(L)}_{i,0}\right) + q_{\text{thr}}\left(S^{(L)}_{i,T} - S^{(L)}_{i,0}\right) \\ &+ \sum_{l=1}^{L-1} \sum_{j_l, j_{l+1}, \ldots, j_{L-1}} C\left(V^{(l)}_{j_l,T} - V^{(l)}_{j_l,0}\right)\left(\prod_{k=l}^{L-2} w^{(k+1)}_{j_k, j_{k+1}}\right) w^{(L)}_{j_{L-1},i} \\ &- \sum_{j_0, j_1, \ldots, j_{L-1}} Q_{j_0,T}\left(\prod_{k=0}^{L-2} w^{(k+1)}_{j_k, j_{k+1}}\right) w^{(L)}_{j_{L-1},i} = 0, \end{aligned} \tag{A35}$$

where $Q_{j_0,T} = \sum_{t=1}^{T} q_{j_0,t}$.

Proof.

We will prove the theorem by mathematical induction on the layer index L.

Base Case ($L = 1$):

For layer $L = 1$, consider neuron i in the first hidden layer. The LoCC for single neuron gives:

$$C\left(V^{(1)}_{i,T} - V^{(1)}_{i,0}\right) + q_{\text{thr}}\left(S^{(1)}_{i,T} - S^{(1)}_{i,0}\right) - Q^{\text{in}}_{i,T} = 0 \tag{A36}$$

where $Q^{\text{in}}_{i,T}$ is the total input charge received by neuron i up to time T.

The input charge $Q^{\text{in}}_{i,T}$ comes from the input layer (layer 0):

$$Q^{\text{in}}_{i,T} = \sum_{t=1}^{T} \sum_{j_0} w^{(1)}_{j_0,i} q_{j_0,t} = \sum_{j_0} Q_{j_0,T} \cdot w^{(1)}_{j_0,j_1} \tag{A37}$$

Substituting back, it matches Equation equation A35 for $L = 1$, as the summation term over l from $l = 1$ to $L - 1 = 0$ is empty (i.e., zero).

Induction Hypothesis:

Assume that the Law of Charge Conservation holds for all neurons in layers up to L ($1 \le L \le L_0 - 1$). That is, equation A35 is satisfied for any neuron in layers 1 through L.

Inductive Step:

We need to show that the Law of Charge Conservation holds for any neuron i in layer $L + 1$.

From LoCC for a single neuron, we have:

$$C\left(V^{(L+1)}_{i,T} - V^{(L+1)}_{i,0}\right) + q_{\text{thr}}\left(S^{(L+1)}_{i,T} - S^{(L+1)}_{i,0}\right) - Q^{\text{in}}_{i,T} = 0 \tag{A38}$$

Now, the total input charge $Q^{\text{in}}_{i,T}$ to neuron i in layer $L+1$ is the sum of the output charges from neurons in layer L weighted by the synaptic weights:

$$Q^{\text{in}}_{i,T} = \sum_{t=1}^{T} \sum_{j_L} w^{(L+1)}_{j_L,i}\, q^{\text{out}}_{j_L,t} \tag{A39}$$

We aim to express the total input charge $Q^{\text{in}}_{i,T}$ in terms of the LoCC equations of the neurons in layer L.

First, note that:

$$\sum_{t=1}^{T} q^{\text{out}}_{j_L,t} = q_{\text{thr}}\left(S^{(L)}_{j_L,T} - S^{(L)}_{j_L,0}\right) \tag{A40}$$

Substituting this into equation A39:

$$Q^{\text{in}}_{i,T} = \sum_{j_L} w^{(L+1)}_{j_L,i}\, q_{\text{thr}}\left(S^{(L)}_{j_L,T} - S^{(L)}_{j_L,0}\right) \tag{A41}$$

With induction hypothesis, we have LoCC holds for the neurons in the L-th layer, so we have:

$$
\begin{aligned}
&Q_{i,T}^{\text{in}} = \sum_{j_L} w_{j_L,i}^{(L+1)} q_{\text{thr}} \left(S_{j_L,T}^{(L)} - S_{j_L,0}^{(L)} \right) \\
&= \sum_{j_L} w_{j_L,i}^{(L+1)} \Big(-C(V_{j_L,T}^{(L)} - V_{j_L,0}^{(L)}) \\
&- \sum_{l=1}^{L-1} \sum_{j_l, j_{l+1}, \cdots, j_{L-1}} C(V_{j_l,T}^{(l)} - V_{j_l,0}^{(l)}) \prod_{k=l}^{L-1} w_{j_k,j_{k+1}}^{(k+1)} \\
&\quad + \sum_{j_0, j_1, \cdots, j_{L-1}} Q_{j_0,T} \prod_{k=0}^{L-1} w_{j_k,j_{k+1}}^{(k+1)} \Big) \\
&= - \sum_{l=1}^{L} \sum_{j_l, j_{l+1}, \cdots, j_L} C(V_{j_l,T}^{(l)} - V_{j_l,0}^{(l)}) \left(\prod_{k=l}^{L-1} w_{j_k,j_{k+1}}^{(k+1)} \right) w_{j_L,i}^{(L+1)} \\
&\quad + \sum_{j_0, j_1, \cdots, j_{L-1}} Q_{j_0,T} \left(\prod_{k=0}^{L-1} w_{j_k,j_{k+1}}^{(k+1)} \right) w_{j_L,i}^{(L+1)}
\end{aligned}
\tag{A42}
$$

Substituting this into equation A38, we can write the Law of Charge Conservation for neuron i in layer $L+1$ as:

$$
\begin{aligned}
&C \left(V_{i,T}^{(L+1)} - V_{i,0}^{(L+1)} \right) + q_{\text{thr}} \left(S_{i,T}^{(L+1)} - S_{i,0}^{(L+1)} \right) \\
&+ \sum_{l=1}^{L} \sum_{j_l, j_{l+1}, \ldots, j_L} C \left(V_{j_l,T}^{(l)} - V_{j_l,0}^{(l)} \right) \left(\prod_{k=l}^{L-1} w_{j_k,j_{k+1}}^{(k+1)} \right) w_{j_L,i}^{(L+1)} \\
&- \sum_{j_0, j_1, \ldots, j_L} Q_{j_0,T} \left(\prod_{k=0}^{L-1} w_{j_k,j_{k+1}}^{(k+1)} \right) w_{j_L,i}^{(L+1)} = 0
\end{aligned}
\tag{A43}
$$

This matches the form of equation A35 for layer $L+1$. Thus, the Law of Charge Conservation holds for neuron i in layer $L+1$.

Conclusion:

By mathematical induction, the Law of Charge Conservation holds for all neurons in the network up to layer L_0. □

Theorem A1.6 (Temporal Independence for a Network) *For an L_0-layer feed-forward ST-BIF neural network with weights θ and no inputs after time step T, the spike tracers of the output neurons at equilibrium, denoted by $\mathbf{S}_{\text{eq}}^{(L_0)} = [S_{i,\text{eq}}^{(L_0)}]_{i=1}^{m_{L_0}}$, are determined by*

$$q_{\text{thr}} \mathbf{S}_{\text{eq}}^{(L_0)} = \mathbf{F}_{\text{QReLU}}(\mathbf{Q}_{\text{total}}; \theta),$$

where the total input charges are $\mathbf{Q}_{\text{total}} = [Q_{j,T}]_{j=1}^{m_0}$.

Remark: This theorem holds under the default settings for S_0, V_0 and $S_{\min}$.

Proof. We will prove the theorem by induction on the layer number L_0.

Base Case ($L_0 = 1$):

For neurons in the 1-th layer (input layer), the theorem reduces to the single-neuron case previously discussed, where the spike tracer $S_{i,\text{eq}}^{(0)}$ and membrane potential $V_{i,\text{eq}}^{(0)}$ are determined solely by $\mathbf{Q}_{\text{total}} = [Q_{j,T}]_{j=1}^{m_0}$ and the network weights θ, independent of input timing.

Inductive Hypothesis:

Assume that the theorem holds for all neural networks with layers $L_0 < L$. So we have

$$q_{\text{thr}} \mathbf{S}_{\text{eq}}^{(L_0)} = \mathbf{F}_{\text{QReLU}}(\mathbf{Q}_{\text{total}}; \theta),$$

for $L_0 < L$.

Inductive Step (Proving for Layer L): From Temporal Independence for a Single Neuron (Theorem A1.4), we have

$$q_{\text{thr}} S_{i,T_{\text{eq}}}^{(L)} = \text{QReLU}(Q_{i,T}^{(L),\text{in}} + \text{bias}_i^{(L)}),$$

with $\text{bias}_i^{(L)}$ is determined by $\theta^{(L)}$, and $Q_{i,T}^{(L),\text{in}}$ is determined by $q_{\text{thr}} \mathbf{S}_{\text{eq}}^{(L-1)}$, which can be calculated as

$$q_{\text{thr}} \mathbf{S}_{\text{eq}}^{(L-1)} = \mathbf{F}_{\text{QReLU}}^{(L-1)}(\mathbf{Q}_{\text{total}}; \theta^{[1:L-1]})$$

from inductive hypothesis. So we have

$$q_{\text{thr}} \mathbf{S}_{\text{eq}}^{(L)} = \mathbf{F}_{\text{QReLU}}(\mathbf{Q}_{\text{total}}; \theta).$$

Conclusion By mathematical induction, the ST-BIF neural network is equivalent to an ANN with a QReLU activation function for any number of layers. Specifically,

$$q_{\text{thr}} \mathbf{S}_{\text{eq}}^{(L_0)} = \mathbf{F}_{\text{QReLU}}(\mathbf{Q}_{\text{total}}; \theta).$$

holds for every positive integer L_0.

□

A1.1.4. Convergence Results

Theorem A1.7 *For an ST-BIF neuron, if external inputs cease after time step T, then the neuron will converge to equilibrium no later than time step $S_{\max} - S_{\min} + T$.*

Proof. After time step T, the neuron no longer receives external inputs. Let V_T and S_T denote the neuron's membrane potential and spike tracer at time T, respectively.

We analyze the neuron's behavior based on its membrane potential at time T:

Case 1: $V_T \geq V_{\text{thr}}$.

- **Subcase 1a:** If $S_T = S_{\max}$, the neuron cannot increase its spike tracer further. Even if $V_T \geq V_{\text{thr}}$, it cannot emit additional positive spikes. Thus, the neuron has already reached equilibrium.

- **Subcase 1b:** If $S_T < S_{\max}$, the neuron will emit positive spikes in subsequent time steps. Each spike reduces the membrane potential by $\frac{q_{\text{thr}}}{C}$. The neuron continues to fire until either:
  1. The membrane potential drops below V_{thr}, or
  2. The spike tracer increases to $S_{\max}$.

  The maximum number of positive spikes the neuron can emit is $S_{\max} - S_T$. Therefore, the neuron will reach equilibrium in at most $S_{\max} - S_T \leq S_{\max} - S_{\min}$ additional time steps.

Case 2: $V_T < 0$.

- **Subcase 2a:** If $S_T = S_{\min}$, the neuron cannot decrease its spike tracer further. Even if $V_T < 0$, it cannot emit additional negative spikes. Thus, the neuron has already reached equilibrium.
- **Subcase 2b:** If $S_T > S_{\min}$, the neuron will emit negative spikes in subsequent time steps. Each spike increases the membrane potential by $\frac{q_{\text{thr}}}{C}$. The neuron continues to fire until either:
  1. The membrane potential rises above 0, or
  2. The spike tracer decreases to $S_{\min}$.

  The maximum number of negative spikes the neuron can emit is $S_T - S_{\min}$. Therefore, the neuron will reach equilibrium in at most $S_T - S_{\min} \leq S_{\max} - S_{\min}$ additional time steps.

In both cases, the neuron will reach equilibrium within $S_{\max} - S_{\min}$ time steps after time T. Therefore, the total time to reach equilibrium is no more than $S_{\max} - S_{\min} + T$ time steps. □

Theorem A1.8 *For an L_0-layer feed-forward ST-BIF neural network, if network inputs cease after time step T, then all neurons converge to equilibrium no later than time step $(S_{\max} - S_{\min})L_0 + T$.*

Proof. By Theorem A1.7, when inputs to the network cease after time step T, all neurons in the first layer stop firing no later than $S_{\max} - S_{\min} + T$. If neurons in the l-th layer stop firing after time step T', then by Theorem A1.7, neurons in the $l+1$-th layer stop firing no later than $S_{\max} - S_{\min} + T'$. Therefore, the L_0-th layer stops firing no later than $(S_{\max} - S_{\min})L_0 + T$. Thus, all neurons in the network converge to equilibrium no later than time step $(S_{\max} - S_{\min})L_0 + T$. □

A1.1.5. Equivalence between DiffEncode and VaniEncode

Theorem A1.9 (Encoding Equivalence) *Consider an L_0-layer feed-forward ST-BIF neural network with weights θ. Under both DiffEncode and VaniEncode, the spike tracers of the output neurons converge to the same equilibrium value. Without loss of generality, for the two input sequences*

$$\mathbf{I}_{\text{diff}} = [\mathbf{X}_0, \mathbf{X}_1 - \mathbf{X}_0, \ldots, \mathbf{X}_k - \mathbf{X}_{k-1}, 0, 0, \ldots, 0]$$

and

$$\mathbf{I}_{\text{vani}} = [\mathbf{X}_k, 0, 0, \ldots, 0],$$

it holds that

$$q_{\text{thr}}\, \mathbf{S}_{\text{eq}}^{(L_0)}(\mathbf{I}_{\text{diff}}) = q_{\text{thr}}\, \mathbf{S}_{\text{eq}}^{(L_0)}(\mathbf{I}_{\text{vani}}).$$

Here, $\mathbf{S}_{\text{eq}}^{(L_0)}(\mathbf{I})$ is the value of spike tracer at equilibrium for input sequence $\mathbf{I}$. Moreover, the network reaches these respective equilibrium values within

$$(S_{\max} - S_{\min})\, L_0 + k + 1 \quad \textit{and} \quad (S_{\max} - S_{\min})\, L_0 + 1$$

time steps, respectively.

Proof. By Theorem A1.6, we have

$$q_{\text{thr}}\, \mathbf{S}_{\text{eq}}^{(L_0)}(\mathbf{I}_{\text{diff}}) = \mathbf{F}_{\text{QReLU}}\big(g(\mathbf{X}_k);\, \theta\big) = q_{\text{thr}}\, \mathbf{S}_{\text{eq}}^{(L_0)}(\mathbf{I}_{\text{vani}}).$$

By Theorem A1.8, these equilibria are reached within

$$(S_{\max} - S_{\min})\, L_i + k + 1 \quad \text{and} \quad (S_{\max} - S_{\min})\, L_i + 1$$

time steps, respectively. Consequently, in finite time, the network converges to the same equilibrium output using either DiffEncode or VaniEncode. □

A1.2. Experiment Setup

In VISTREAM, we first train a quantized Quantized backbone and convert the QANN to SNN. We choose pretrained ResNet-50 as the backbone of Unitrack and YOLOV3. Then, we insert quantize functions to ResNet-50 and train the QANN for 90 epochs, with 1.5e-4 base learning rate, 1024 batch size, 0.05 weight decay, cosine learning rate schedule, and several data augments including mixup [40] and auto-augment [5].

A1.3. Detailed Energy Analysis

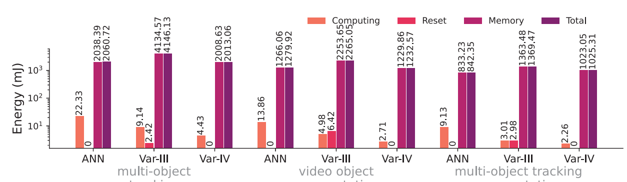

Figure A1. Energy analysis containing data movement energy.

We further analyze the energy using the model in [24] that counts the data movement, whose results are shown in A1. Results show that most of the energy is consumed by data movement. By using the *DiffEncode*, VISTREAM achieves an overall energy comparable to ANN without sacrificing the equivalence of QANN. Moreover, by directly using the compress method proposed in [38], which is already part of ViStream, we reduced the energy from 2006.85mJ to 1382.88mJ without accuracy loss in multi-object tracking tasks, further amplifying the efficiency gains.

A1.4. Analysis of SOP and Input Magnitude under Varying Frame Rates

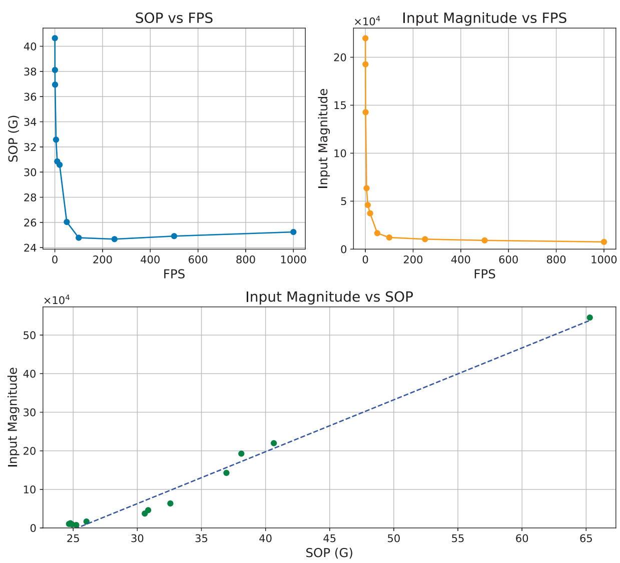

Figure A2. SOP and input magnitude across varying frame rates (top) and their correlation (bottom), measured on CARLA-simulated data with YOLO V3 under DiffEncode.

In this appendix, we provide additional details on the CARLA-based experiments to facilitate reproducibility and clarify the setup used to evaluate the ViStream framework in conjunction with the YOLO V3 network under the Diff-Encode paradigm. We employed CARLA (version 0.9.13) and loaded the `Town03` map in synchronous mode, enabling precise frame-by-frame control of the simulation. A Tesla Model 3 blueprint was randomly spawned on the map, and its autopilot functionality was managed via the Traffic Manager with all traffic signals and speed limits ignored to maintain a continuous driving sequence. We set the camera resolution to 416×416 pixels and attached the sensor at a fixed offset (approximately 1.5 m forward and 2.4 m above the vehicle's center).

For each run, we collected a total of 3,000 frames, adjusting the simulator's fixed time-step to achieve the target frame rates. Specifically, we tested frame rates of 1 – 1,000 Hz to cover both typical and ultra-high regimes of modern CMOS image sensors. These images were then fed into ViStream to measure the corresponding spiking operations per frame (SOP) and input magnitude.

As shown in Figure A2, within the usual range of 60 Hz to 100 Hz, moderate increases in the frame rate significantly reduce both SOP and input magnitude, leading to improved energy efficiency. However, once the frame rate escalates to several hundred hertz and beyond, we observe diminishing returns where further rate increments yield only marginal decreases in SOP and input magnitude. In future work, we plan to explore more sophisticated encoding strategies and alternative sensor processing methods to maintain the energy advantages of ViStream across a broader range of frame rates, including the ultra-high-FPS regime.