Stable Mean Flow: Lyapunov-Inspired One-Step Flow Matching

Supplementary Material

7. Rationale

7.1. Proof of Theorem 3.1

Proof. Fix $r \in [0,1)$ and $t \in (r,1]$, and set $\alpha := t - r > 0$. By non–expansivity, for any base point z_t and any perturbation Δz with $\|\Delta z\| \le \delta$ we have

$$\begin{aligned}\big\|\Delta z - \alpha\big(u_\theta(z_t + \Delta z, r, t) - u_\theta(z_t, r, t)\big)\big\| \\ = \big\|\phi_r^\theta(t, z_t + \Delta z) - \phi_r^\theta(t, z_t)\big\|_2 \\ \le \|\Delta z\|\end{aligned}$$

If we define

$$\Delta u := u_\theta(z_t + \Delta z, r, t) - u_\theta(z_t, r, t)$$

then the triangle inequality yields

$$\|\alpha \Delta u\| \le \|\Delta z\| + \|\Delta z - \alpha \Delta u\| \le 2\|\Delta z\|.$$

Therefore

$$\|u_\theta(z_t + \Delta z, r, t) - u_\theta(z_t, r, t)\| = \|\Delta u\| \le \frac{2}{\alpha}\,\|\Delta z\|. \tag{1}$$

Now fix $t_0 \in (r,1]$ and consider any compact time interval $I \subseteq [t_0, 1]$. For every $t \in I$ we have $t - r \ge t_0 - r > 0$, hence

$$\frac{2}{\alpha} = \frac{2}{t - r} \le \frac{2}{t_0 - r} \qquad \text{for all } t \in I.$$

Thus $u_\theta(\cdot, r, t)$ is locally Lipschitz in z with a uniform Lipschitz bound on I. By construction of the Stable Mean Flow field, u_θ is also continuous in t.

Consequently, u_θ is continuous and locally Lipschitz in z with a uniform Lipschitz constant. By the Picard–Lindelöf theorem, for any initial condition (t_0, z_0) there exists a unique solution $z(\cdot)$ of

$$\dot z(s) = u_\theta\big(z(s), r, s\big), \qquad z(t_0) = z_0,$$

on some interval containing t_0 within $[t_0, 1]$. Since the above assumptions hold on the whole strip $K \times [t_0, 1]$, this solution extends uniquely throughout any subinterval of $[t_0, 1]$.

Hence, every initial condition (t_0, z_0) with $t_0 \in (r, 1]$ determines a unique trajectory $z(\cdot)$, so characteristics of the learned Stable Mean Flow field cannot cross or branch. □

7.2. Proof of Theorem 3.2

Proof. The Jacobian vector product (JVP) formula is

$$\begin{aligned}\frac{d}{dt}u(z_t, r, t) &= \underbrace{\nabla_z u(z_t, r, t)}_{\partial_z u}\cdot \underbrace{\frac{dz_t}{dt}}_{u(z_t, r, t)} + \underbrace{\frac{\partial u(z_t, r, t)}{\partial t}}_{\text{time partial}} \\ &= (\partial_z u)\cdot u(z_t, r, t) + \partial_t u(z_t, r, t).\end{aligned}$$

In addition, from the Lipschitz bound 1, we get

$$\|u_\theta(z_t + \Delta z, r, t) - u_\theta(z_t, r, t)\| \;\le\; \frac{2}{\alpha}\,\|\Delta z\|.$$

Divide by $\|\Delta z\|$ and take $\Delta z \to 0$. Since $u_\theta(\cdot, r, t)$ is C^1 in z, we then have

$$\|\partial_z u_\theta(z_t, r, t)\| \;\le\; \frac{2}{\alpha}.$$

One of the working assumptions is that the neural network representing u_θ is C^1 and defined on a compact set of parameters θ. Hence $\|u_\theta\| \le M_\theta$ and $\|\partial_t u_\theta\| \le \Lambda_\theta$ for some constants $M_\theta, \Lambda_\theta < \infty$. This shows that the JVP is bounded by a constant

$$\big\|(\partial_z u_\theta)\cdot u_\theta + \partial_t u_\theta\big\| \;\le\; \frac{2}{\alpha}M_\theta + \Lambda_\theta \le C.$$

□

7.3. Proof of Theorem 3.3

Proof. From the definition of ϕ_r^θ, write

$$V(\Delta z) = \|\Delta z - \alpha\,\Delta u\|_2 - \|\Delta z\|_2,$$

where

$$\Delta u := u_\theta(z_t + \Delta z, r, t) - u_\theta(z_t, r, t).$$

By the triangle inequality, for any vectors a, b,

$$\|a + b\| - \|a\| \le \|b\|.$$

Applying this with $a = \Delta z$ and $b = -\alpha\,\Delta u$ yields the pointwise bound

$$V(\Delta z) \;\le\; \alpha\,\|\Delta u\|_2.$$

Since $\|\Delta z\|_2 \le \delta$ almost surely when selected from the distribution $\mathcal{D}_\delta$, the local Lipschitz bound (1) gives

$$\|\Delta u\|_2 \;\le\; \frac{2}{\alpha}\,\|\Delta z\|_2,$$

and therefore

$$V(\Delta z) \;\le\; \alpha\cdot\frac{2}{\alpha}\,\|\Delta z\|_2 \;=\; 2\,\|\Delta z\|_2 \qquad \text{(a.s.)}.$$

Taking expectations gives

$$\mathbb{E}\,[V(\Delta z)] \;\le\; 2\,\mathbb{E}\|\Delta z\|_2 \;\le\; 2\,\delta.$$

For the tail probability, apply Markov's inequality to $V(\Delta z)$:

$$\mathbb{P}(V(\Delta z) > \tau) \;\le\; \frac{\mathbb{E}[V(\Delta z)]}{\tau} \;\le\; \frac{2\,\delta}{\tau} \qquad (\tau > 0).$$

This completes the proof. □

7.4. Proof of Theorem 4.1

Proof. Write $\alpha_t := t - r$ and $\alpha_{t+\Delta t} := t + \Delta t - r$. From the oracle identity at $s = t + \Delta t$ and $s = t$ we get

$$\alpha_{t+\Delta t}\,u^*(t + \Delta t, z_{t+\Delta t}) - \alpha_t\,u^*(t, z_t) \;=\; z_{t+\Delta t} - z_t.$$

Hence

$$\begin{aligned}&\alpha_{t+\Delta t}e_{t+\Delta t} - \alpha_t e_t \\ &= \alpha_{t+\Delta t}u_\theta(z_{t+\Delta t}, r, t + \Delta t) - \alpha_t u_\theta(z_t, r, t) - \big(z_{t+\Delta t} - z_t\big).\end{aligned}$$

Add and subtract $\alpha_{t+\Delta t} u_\theta(z_t, r, t + \Delta t)$, so that $\alpha_{t+\Delta t} e_{t+\Delta t} - \alpha_t e_t$ becomes

$$\underbrace{\Big(\alpha_{t+\Delta t} u_\theta(z_{t+\Delta t}, r, t+\Delta t) - \alpha_{t+\Delta t} u_\theta(z_t, r, t+\Delta t)\Big)}_{\text{(I)}} + \underbrace{\Big(\alpha_{t+\Delta t} u_\theta(z_t, r, t+\Delta t) - \alpha_t u_\theta(z_t, r, t)\Big)}_{\text{(II)}} - (z_{t+\Delta t} - z_t).$$

To handle (I), observe

$$\text{(I)} - (z_{t+\Delta t} - z_t) = -\Big(\phi_r^\theta(t+\Delta t, z_{t+\Delta t}) - \phi_r^\theta(t+\Delta t, z_t)\Big),$$

so by non-expansivity of ϕ_r^θ in z,

$$\|\text{(I)} - (z_{t+\Delta t} - z_t)\| \;\le\; \|z_{t+\Delta t} - z_t\|.$$

For (II), decompose $\alpha_{t+\Delta t} = \Delta t + \alpha_t$ to write:

$$\text{(II)} = \underbrace{\Delta t\, u_\theta(z_t, r, t+\Delta t)}_{\text{bounded by } \Delta t\, M_\theta} + \alpha_t \underbrace{\big(u_\theta(z_t, r, t+\Delta t) - u_\theta(z_t, r, t)\big)}_{\text{time diff.}}$$

and, by the fundamental theorem of calculus with $\partial_s u_\theta$ and the assumed bound $\|\partial_s u_\theta\| \le \Lambda_\theta$, we find

$$\begin{aligned}\|u_\theta(z_t, r, t+\Delta t) - u_\theta(z_t, r, t)\| &= \left\|\int_t^{t+\Delta t} \partial_s u_\theta(z_t, r, s)\, ds\right\| \\ &\le \int_t^{t+\Delta t} \Lambda_\theta\, ds = \Delta t\, \Lambda_\theta.\end{aligned}$$

Therefore using the assumption $\|u_\theta\| \le M_\theta$,

$$\|\text{(II)}\| \;\le\; \Delta t\, M_\theta \;+\; \alpha_t\, \Delta t\, \Lambda_\theta.$$

Combine them and apply the triangle inequality:

$$\big\|\alpha_{t+\Delta t} e_{t+\Delta t} - \alpha_t e_t\big\| \;\le\; \|z_{t+\Delta t} - z_t\| \;+\; \Delta t\Big(M_\theta + \alpha_t \Lambda_\theta\Big).$$

Since $\alpha_t = t - r$. Finally, we obtain the result by dividing both sides by $\alpha_{t+\Delta t} = t + \Delta t - r > 0$. □

7.5. Proof of Corollary 4.1

Proof. Following from the above theorem, we then have

$$\begin{aligned}\|e_{t_{k+1}}\| &\le \frac{\alpha_{t_k}}{\alpha_{t_k} + \Delta t_k} \|e_{t_k}\| \;+\; \frac{\Delta t_k}{\alpha_{t_k} + \Delta t_k}\Big(M^* + M_\theta + \alpha_{t_k}\Lambda_\theta\Big) \\ &= (1 - \beta_k)\, \|e_{t_k}\| \;+\; \beta_k\, T_{t_k}.\end{aligned}$$

We consider cases $\|e_{t_k}\| \le T_{t_k}$ and $\|e_{t_k}\| > T_{t_k}$.

1. In the first case, $\|e_{t_{k+1}}\| \le (1-\beta_k) T_{t_k} + \beta_k T_{t_k} = T_{t_k}$.
2. In the second case, it directly follows

$$\|e_{t_{k+1}}\| = \|e_{t_k}\| - \beta_k\big(\|e_{t_k}\| - T_{t_k}\big) < \|e_{t_k}\| \quad (\beta_k > 0),$$

So we find $\|e_{t_{k+1}}\| \;\le\; \max\{\|e_{t_k}\|,\, T_{t_k}\}$. □

7.6. Proof of Corollary 4.2

Proof. For each step $t_k \to t_{k+1}$ with $h_k := t_{k+1} - t_k$ the forward one–step bound in Theorem 4.1 gives

$$\big\|\alpha_{t_{k+1}} e_{t_{k+1}} - \alpha_{t_k} e_{t_k}\big\| \;\le\; \|z_{t_{k+1}} - z_{t_k}\| + h_k\big(M_\theta + \alpha_{t_k}\Lambda_\theta\big).$$

and hence

$$\big\|\alpha_{t_{k+1}} e_{t_{k+1}} - \alpha_{t_k} e_{t_k}\big\| \;\le\; h_k\Big(M^* + M_\theta + \alpha_{t_k}\Lambda_\theta\Big).$$

Summing $k = 0, \dots, N-1$ and telescoping yields

$$\|\alpha_1 e_1 - \alpha_t e_t\| \;\le\; \sum_{k=0}^{N-1} h_k\Big(M^* + M_\theta + \alpha_{t_k}\Lambda_\theta\Big).$$

Taking the mesh $\max h_k \to 0$ gives the integral form

$$\alpha_1\|e_1\| \;\le\; \alpha_t\|e_t\| + \int_t^1 \big(M^* + M_\theta + \alpha_s \Lambda_\theta\big)\, ds.$$

We can compute these explicity:

$$\int_t^1 (M^* + M_\theta)\, ds = (1-t)(M^* + M_\theta)$$

$$\int_t^1 \alpha_s\, ds = \tfrac{1}{2}\big(\alpha_1^2 - \alpha_t^2\big).$$

Hence

$$\alpha_1\|e_1\| \;\le\; \alpha_t\|e_t\| + (1-t)(M^* + M_\theta) + \tfrac{1}{2}\big(\alpha_1^2 - \alpha_t^2\big)\Lambda_\theta.$$

Then, we divide $\alpha_1 = 1 - r$. For $t = 1 - \eta$, the last two terms are $O(\eta)$, giving $\|e_1\| \le \|e_{1-\eta}\| + O(\eta)$. □