1. Real Object Dataset

We illustrate our real object dataset at the end of this supplemental material (depth maps and masks). We place an Intel L515 packing box (115mm × 115mm × 60mm) under each item as a size reference.

2. Mathematical Definition and Theorem

For self-contained, we give the important definition (Wasserstein space) and theorem (Displacement interpolation) in the following subsections. It may help readers to understand our unique displacement interpolation framework.

2.1. Wasserstein Space

Definition 1 Wasserstein space \[ W_p(X) := \{ \mu \in P(X); \int_X d(x_0, x)^p \mu(dx) < +\infty \} \]

where \( \mu \) is probability measures on \( X \), and \( x_0 \in X \) is arbitrary. This space does not depend on the choice of the point \( x_0 \). Then \( W_p \) defines a (finite) distance on \( P_p(X) \).

2.2. Displacement Interpolation

Theorem 2.1 Displacement interpolation \[ \gamma \]

Let \( \gamma \) be a Polish space, and \( (\mathcal{A})^{0.1} \) a coercive Lagrangian action on \( X \), with continuous cost functions \( c_{s,t} \). Whenever \( 0 \leq s < t \leq 1 \), denote by \( C_{s,t}^{\mu,\nu} \) the optimal transport cost between the probability measures \( \mu \) and \( \nu \) for the cost \( c_{s,t} \); write \( c = c_{0,1} \) and \( C = C_{0,1} \). Let \( \mu_0 \) and \( \mu_1 \) be any two probability measures on \( X \), such that the optimal transport cost \( C(\mu_0, \mu_1) \) is finite. Then, given a continuous path \( (\mu_t)_{0\leq t\leq 1} \), the following properties are equivalent:

(i) For each \( t \in [0, 1] \), \( \mu_t \) is the law of \( \gamma_t \), where \( (\gamma_t)_{0\leq t\leq 1} \) is a dynamical optimal coupling of \( (\mu_0, \mu_1) \);

(ii) For any of three intermediate times \( t_1 < t_2 < t_3 \) in \( [0, 1] \),

\[ C^{t_1, t_2}(\mu_{t_1}, \mu_{t_2}) + C^{t_2, t_3}(\mu_{t_2}, \mu_{t_3}) = C^{t_1, t_3}(\mu_{t_1}, \mu_{t_3}) \]

(iii) The path \( (\mu_t)_{0\leq t\leq 1} \) is a minimizing curve for the coercive action functional defined on \( P(X) \) by

\[
A_{s,t}^{\gamma}(\mu) = \sup_{N \in \mathbb{N}} \sup_{s = t_0 < t_1 < \ldots < t_N = t} \sum_{i=0}^{N-1} C^{t_i, t_{i+1}}(\mu_{t_i}, \mu_{t_{i+1}})
= \inf_{\gamma} E_\gamma A_{s,t}^{\gamma},
\]

where the last infimum is over all random curves \( \gamma : [s, t] \to X \) such that law \( (\gamma_t) = \mu_t(s \leq t) \).

Corollary 1 Uniqueness of displacement interpolation \[ \gamma \]

With the same assumptions as in Theorem Displacement interpolation, if:

(i) there is a unique optimal transport plan \( \pi \) between \( \mu_0 \) and \( \mu_1 \);

(ii) \( \pi(dx_0, dx_1) \) almost surely, \( x_0 \) and \( x_1 \) are joined by a unique minimizing curve;

then there is a unique displacement interpolation \( (\mu_t)_{0\leq t\leq 1} \) joining \( \mu_0 \) to \( \mu_1 \).

References
