GeomNet: A Neural Network Based on Riemannian Geometries of SPD Matrix Space and Cholesky Space for 3D Skeleton-Based Interaction Recognition

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This supplementary material provides the proofs for the Theorems and Lemmas presented in the paper. It also gives more details on the datasets and experimental protocols used in our experiments. We refer the interested reader to [69, 70] for an introduction to the theory of Lie groups and Riemannian symmetric spaces. Please also see the paper for references.

1. Proof of Lemma 1

Proof. We apply the tricks in [34]. Let $\mathbb{R}^{n\times m}$ be the set of $n\times m$ matrices. The following notations will be used in the material:

$$\begin{aligned} GL(n) &= \{\mathbf{A}\in\mathbb{R}^{n\times n} |\det(\mathbf{A})\neq 0\},\\ GL^+(n) &= \{\mathbf{A}\in GL(n)|\det(\mathbf{A})>0\},\\ SL(n) &= \{\mathbf{A}\in GL(n)|\det(\mathbf{A})=1\},\\ O(n) &= \{\mathbf{A}\in GL(n)|\mathbf{A}^T=\mathbf{A}^{-1}\},\\ SO(n) &= \{\mathbf{A}\in O(n)|\det(\mathbf{A})=1\}. \end{aligned} \tag{35}$$

Denote by $Aff(n)$ the affine group of $\mathbb{R}^n$ which is the semidirect product:

$$Aff(n) = \mathbb{R}^n \rtimes GL(n), \tag{36}$$

where the semidirect product is defined as $(\mathbf{A}_1,\mathbf{b}_1).(\mathbf{A}_2,\mathbf{b}_2) = (\mathbf{A}_1\mathbf{A}_2, \mathbf{b}_1+\mathbf{A}_1\mathbf{b}_2)$ for $(\mathbf{A}_i,\mathbf{b}_i)\in Aff(n), i=1,2$.

Considering the action of affine group $Aff(n)$ on $N(n)$:

$$(\mathbf{A},\mathbf{b}).(\boldsymbol{\Sigma},\boldsymbol{\mu}) = (\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T, \mathbf{A}\boldsymbol{\mu}+\mathbf{b}), \tag{37}$$

where $\mathbf{A}\in GL(n)$, $\mathbf{b}\in\mathbb{R}^n$, and $(\boldsymbol{\Sigma},\boldsymbol{\mu})\in N(n)$.

It is easy to show that this action is transitive. The projection of $Aff(n)$ onto $N(n)$ is:

$$\begin{aligned} &\pi_1 : Aff(n)\to N(n),\\ &(\mathbf{A},\mathbf{b})\mapsto (\mathbf{A},\mathbf{b}).(\mathbf{I}_n,\mathbf{0}_{n\times 1}) = (\mathbf{A}\mathbf{A}^T,\mathbf{b}). \end{aligned} \tag{38}$$

The stabilizer of the standard Gaussian is given by:

$$\mathbf{A}\mathbf{0}_{n\times 1}+\mathbf{b} = \mathbf{0}_{n\times 1}, \text{ and } \mathbf{A}\mathbf{I}_n\mathbf{A}^T = \mathbf{I}_n. \tag{39}$$

Therefore, the stabilizer of the standard Gaussian is $O(n)$ and we have: $Aff(n)/O(n)\cong N(n)$. Let $Aff^+(n)$ be the semidirect product:

$$Aff^+(n) = \mathbb{R}^n \rtimes GL^+(n). \tag{40}$$

Then the action remains transitive when restricted to $Aff^+(n)$. From (39) we deduce that $\mathbf{A}\mathbf{A}^T = \mathbf{I}_n$ and $\det\mathbf{A}=1$. Therefore, the stabilizer of the standard Gaussian is $SO(n)$. Consequently, we get: $Aff^+(n)/SO(n)\cong N(n)$.

Now, we embed $Aff^+(n)$ into the Lie group $SL(n+k)$ as follows:

$$\begin{aligned} &Aff^+(n)\hookrightarrow SL(n+k),\\ &(\mathbf{A},\mathbf{b})\mapsto (\det\mathbf{A})^{-\frac{1}{n+k}}\begin{bmatrix}\mathbf{A} & \mathbf{b}(k)\\ \mathbf{0}_{k\times n} & \mathbf{I}_k\end{bmatrix}. \end{aligned} \tag{41}$$

The group $SL(n+k)$ acts on $Sym^{+,1}_{n+k}$ as follows: For all $\mathbf{U}\in SL(n+k)$ and all $\mathbf{P}\in Sym^{+,1}_{n+k}$:

$$\mathbf{U}.\mathbf{P} = \mathbf{U}\mathbf{P}\mathbf{U}^T. \tag{42}$$

It is easily checked that $\mathbf{U}\mathbf{P}\mathbf{U}^T\in Sym^{+,1}_{n+k}$ if $\mathbf{P}\in Sym^{+,1}_{n+k}$. Since every $\mathbf{P}\in Sym^{+,1}_{n+k}$ can be written as $\mathbf{P}=\mathbf{U}\mathbf{U}^T$ for some $\mathbf{U}\in SL(n+k)$, the action is transitive. It is also easy to see that the stabilizer of the identity element in $Sym^{+,1}_{n+k}$ is $SO(n+k)$. Then we can conclude that $SL(n+k)/SO(n+k)\cong Sym^{+,1}_{n+k}$. The projection of $SL(n+k)$ onto $Sym^{+,1}_{n+k}$ is:

$$\begin{aligned} &\pi_2 : SL(n+k)\to Sym^{+,1}_{n+k},\\ &\mathbf{U}\mapsto \mathbf{U}\mathbf{U}^T. \end{aligned} \tag{43}$$

Combining with (41), the restriction of π_2 to the subgroup $Aff^+(n)$ is given by:

$$\begin{aligned} &Aff^+(n)\to Sym^{+,1}_{n+k},\\ &(\mathbf{A},\mathbf{b})\mapsto (\det\mathbf{A})^{-\frac{2}{n+k}}\begin{bmatrix}\mathbf{A}\mathbf{A}^T+\mathbf{b}(k)\mathbf{b}(k)^T & \mathbf{b}(k)\\ \mathbf{b}(k)^T & \mathbf{I}_k\end{bmatrix}. \end{aligned} \tag{44}$$

This restriction is a surjective mapping with $\pi_2^{-1}(\mathbf{I}_{n+k}) = SO(n)$. Note that we have the following embedding:

$$\begin{aligned} &SO(n) \hookrightarrow SO(n+k), \\ &\mathbf{A} \mapsto \begin{bmatrix} \mathbf{A} & \mathbf{0}_{n\times k} \\ \mathbf{0}_{k\times n} & \mathbf{I}_k \end{bmatrix}. \end{aligned} \tag{45}$$

We can now conclude that $N(n)$ can be identified with the symmetric space $Sym_{n+k}^{+,1} \cong SL(n+k)/SO(n+k)$. Specifically, we have the following embedding:

$$\begin{aligned} &\bar{\psi} : N(n) \to Sym_{n+k}^{+,1} \\ &(\mathbf{A}\mathbf{A}^T, \mathbf{b}) \mapsto (\det \mathbf{A})^{-\frac{2}{n+k}} \begin{bmatrix} \mathbf{A}\mathbf{A}^T + \mathbf{b}(k)\mathbf{b}(k)^T & \mathbf{b}(k) \\ \mathbf{b}(k)^T & \mathbf{I}_k \end{bmatrix}. \end{aligned} \tag{46}$$

We can change the coordinates and obtain an embedding of the Gaussian $(\boldsymbol{\Sigma}, \boldsymbol{\mu}) \in N(n)$:

$$\begin{aligned} &\psi : N(n) \to Sym_{n+k}^{+,1} \\ &(\boldsymbol{\Sigma}, \boldsymbol{\mu}) \mapsto (\det \boldsymbol{\Sigma})^{-\frac{1}{n+k}} \begin{bmatrix} \boldsymbol{\Sigma} + k\boldsymbol{\mu}\boldsymbol{\mu}^T & \boldsymbol{\mu}(k) \\ \boldsymbol{\mu}(k)^T & \mathbf{I}_k \end{bmatrix}. \end{aligned} \tag{47}$$

□

2. Proof of Lemma 2

Proof. Let $gl(n)$, $sl(n)$, and $aff(n)$ be the Lie algebras of the Lie groups $GL(n)$, $SL(n)$, and $Aff(n)$, respectively. It has been known that:

$$\begin{aligned} gl(n) &= \{\mathbf{A} | \mathbf{A} \in \mathbb{R}^{n\times n}\}, \\ sl(n) &= \{\mathbf{A} | \mathbf{A} \in gl(n), Tr(\mathbf{A}) = 0\}, \\ aff(n) &= \{(\mathbf{A}, \mathbf{b}) | \mathbf{A} \in gl(n), \mathbf{b} \in \mathbb{R}^n\}. \end{aligned} \tag{48}$$

Denote by $T_{\mathbf{I}_{n+k}} Sym_{n+k}^{+,1}$ the tangent space at the identity of $Sym_{n+k}^{+,1}$, $T_{(\mathbf{I}_n, \mathbf{0}_{n\times 1})} N(n)$ the tangent space at the identity of $N(n)$. These spaces can be identified by Cartan decomposition. We have the following proposition.

Proposition 1. *$T_{\mathbf{I}_{n+k}} Sym_{n+k}^{+,1}$ is identified with:*

$$\{\mathbf{A} | \mathbf{A} \in gl(n+k), Tr(\mathbf{A}) = 0, \mathbf{A} = \mathbf{A}^T\}. \tag{49}$$

$T_{(\mathbf{I}_n, \mathbf{0}_{n\times 1})} N(n)$ is identified with:

$$\{(\mathbf{A}, \mathbf{b}) | \mathbf{A} \in gl(n), \mathbf{A} = \mathbf{A}^T, \mathbf{b} \in \mathbb{R}^n\}. \tag{50}$$

Proof. We will prove the first part of Proposition 1. The proof of the second part can be done in the same way. Based on Lemma 1, we know that $Sym_{n+k}^{+,1} \cong SL(n+k)/SO(n+k)$. An involution on $SL(n+k)$ is given by: $\sigma(\mathbf{U}) = (\mathbf{U}^{-1})^T$. Then the Cartan involution is computed as:

$$\Theta(\mathbf{A}) = \frac{d}{dt}\bigg|_{t=0} \sigma(\exp(t\mathbf{A})) = \frac{d}{dt}\bigg|_{t=0} \exp(-t\mathbf{A}^T) = -\mathbf{A}^T. \tag{51}$$

Thus, $sl(n+k)$ decomposes as a vector space as $sl(n+k) = \mathfrak{l} \oplus \mathfrak{m}$ where $\mathfrak{l}$ and $\mathfrak{m}$ are the eigenspaces of Θ for the eigenvalues 1 and -1, respectively:

$$\begin{aligned} \mathfrak{l} &= \{\mathbf{A} | \mathbf{A} \in gl(n+k), Tr(\mathbf{A}) = 0, \Theta(\mathbf{A}) = \mathbf{A}\} \\ &= \{\mathbf{A} | \mathbf{A} \in gl(n+k), Tr(\mathbf{A}) = 0, \mathbf{A} + \mathbf{A}^T = 0\}, \end{aligned} \tag{52}$$

$$\begin{aligned} \mathfrak{m} &= \{\mathbf{A} | \mathbf{A} \in gl(n+k), Tr(\mathbf{A}) = 0, \Theta(\mathbf{A}) = -\mathbf{A}\} \\ &= \{\mathbf{A} | \mathbf{A} \in gl(n+k), Tr(\mathbf{A}) = 0, \mathbf{A} = \mathbf{A}^T\}. \end{aligned} \tag{53}$$

Therefore, we can identify $\mathfrak{m}$ with the tangent space $T_{\mathbf{I}_{n+k}} Sym_{n+k}^{+,1}$. □

Proposition 2. *The differential $d_{(\mathbf{I}_n, \mathbf{0}_{n\times 1})}\psi : T_{(\mathbf{I}_n, \mathbf{0}_{n\times 1})} N(n) \to T_{\mathbf{I}_{n+k}} Sym_{n+k}^{+,1}$ of ψ at $(\mathbf{I}_n, \mathbf{0}_{n\times 1})$ is given by:*

$$(d_{(\mathbf{I}_n, \mathbf{0}_{n\times 1})}\psi)(\mathbf{A}, \mathbf{b}) = \begin{bmatrix} \mathbf{A} - \frac{1}{n+k} Tr(\mathbf{A})\mathbf{I}_n & \mathbf{b}(k) \\ \mathbf{b}(k)^T & -\frac{1}{n+k} Tr(\mathbf{A})\mathbf{I}_k \end{bmatrix}. \tag{54}$$

Proof. Let $\mathbf{B} = \mathbf{b}(k)$, $\mathbf{D} = D_1.\mathbf{D}_2$ and $D_1, \mathbf{D}_2$ are given by :

$$D_1 = \det(\mathbf{I}_n + t\mathbf{A})^{-\frac{1}{n+k}}, \tag{55}$$

$$\mathbf{D}_2 = \begin{bmatrix} \mathbf{I}_n + t\mathbf{A} + t\mathbf{B}t\mathbf{B}^T & t\mathbf{B} \\ t\mathbf{B}^T & \mathbf{I}_k \end{bmatrix}. \tag{56}$$

Note that $\frac{d\mathbf{D}}{dt} = \frac{dD_1}{dt}\mathbf{D}_2 + D_1\frac{d\mathbf{D}_2}{dt}$ and the differential $\frac{dD_1}{dt}$ is given by:

$$-\frac{1}{n+k} \det(\mathbf{I}_n + t\mathbf{A})^{-\frac{n+k+1}{n+k}} \det(\mathbf{I}_n + t\mathbf{A}) Tr((\mathbf{I}_n + t\mathbf{A})^{-1}\mathbf{A}). \tag{57}$$

The differential $\frac{d\mathbf{D}_2}{dt}$ is given by:

$$\begin{bmatrix} \mathbf{A} + 2t\mathbf{B}\mathbf{B}^T & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0}_{k\times k} \end{bmatrix}. \tag{58}$$

Hence, we obtain $\frac{d\mathbf{D}}{dt}\Big|_{t=0}$ as:

$$-\frac{1}{n+k} Tr(\mathbf{A}) \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times k} \\ \mathbf{0}_{k\times n} & \mathbf{I}_k \end{bmatrix} + \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0}_{k\times k} \end{bmatrix}, \tag{59}$$

which can be compactly written as:

$$\begin{bmatrix} \mathbf{A} - \frac{1}{n+k} Tr(\mathbf{A})\mathbf{I}_n & \mathbf{B} \\ \mathbf{B}^T & -\frac{1}{n+k} Tr(\mathbf{A})\mathbf{I}_k \end{bmatrix}. \tag{60}$$

□

We are now ready to prove Lemma 2. It has been known that the natural $SL(n+k)-$invariant metric on $Sym^{+,1}_{n+k}$ is, up to a positive multiple, given by the Killing form of the Lie algebra $sl(n+k) = \mathfrak{l} \oplus \mathfrak{m}$, i.e.:

$$< \mathbf{U}_1, \mathbf{U}_2 >= Tr(\mathbf{U}_1\mathbf{U}_2),\ \mathbf{U}_1, \mathbf{U}_2 \in T_{\mathbf{I}_{n+k}} Sym^{+,1}_{n+k}. \tag{61}$$

From Proposition 2, it follows that the symmetric metric on $T_{(\mathbf{I}_n, \mathbf{0}_{n\times 1})} N(n)$ is given by:

$$\begin{aligned} < (\mathbf{A}_1, \mathbf{b}_1), (\mathbf{A}_2, \mathbf{b}_2) >= & Tr(\mathbf{A}_1\mathbf{A}_2) + 2Tr(\mathbf{b}_1^T \mathbf{b}_2) \\ & - \frac{1}{n+k} Tr(\mathbf{A}_1) Tr(\mathbf{A}_2). \end{aligned} \tag{62}$$

Thus, we have:

$$\begin{aligned} < (\mathbf{A}_1, \mathbf{0}_{n\times 1}), (\mathbf{A}_2, \mathbf{0}_{n\times 1}) >= & Tr(\mathbf{A}_1\mathbf{A}_2) \\ & - \frac{1}{n+k} Tr(\mathbf{A}_1) Tr(\mathbf{A}_2). \end{aligned} \tag{63}$$

The metric at any point $\mathbf{P}$ is obtained by transporting (63) by the action of the affine group:

$$\begin{aligned} < \mathbf{A}_1, \mathbf{A}_2 >_{\mathbf{P}} = & Tr(\mathbf{A}_1\mathbf{P}^{-1}\mathbf{A}_2\mathbf{P}^{-1}) - \\ & - \frac{1}{n+k} Tr(\mathbf{A}_1\mathbf{P}^{-1}) Tr(\mathbf{A}_2\mathbf{P}^{-1}). \end{aligned} \tag{64}$$

□

3. Proof of Lemma 3

Proof. The proof is based on [11, 61]. Let $\mathbf{P}, \mathbf{Q} \in Sym^+_n$, $\mathbf{A}_1, \mathbf{A}_2 \in T_{\mathbf{Q}} Sym^+_n$. Denote by $\mathbf{E} = (\mathbf{P}\mathbf{Q}^{-1})^{\frac{1}{2}} = \mathbf{Q}^{\frac{1}{2}} \left(\mathbf{Q}^{-\frac{1}{2}}\mathbf{P}\mathbf{Q}^{-\frac{1}{2}}\right)^{\frac{1}{2}} \mathbf{Q}^{-\frac{1}{2}}$. Note that $\mathbf{P}^{-1}\mathbf{E}$ is a symmetric matrix since:

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{E} &= \mathbf{P}^{-1}\mathbf{Q}^{\frac{1}{2}} \left(\mathbf{Q}^{-\frac{1}{2}}\mathbf{P}\mathbf{Q}^{-\frac{1}{2}}\right)^{\frac{1}{2}} \mathbf{Q}^{-\frac{1}{2}} \\ &= \mathbf{P}^{-1}\mathbf{Q}^{\frac{1}{2}}\mathbf{Q}^{-\frac{1}{2}}\mathbf{P}\mathbf{Q}^{-\frac{1}{2}} \left(\mathbf{Q}^{-\frac{1}{2}}\mathbf{P}\mathbf{Q}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \mathbf{Q}^{-\frac{1}{2}} \\ &= \mathbf{Q}^{-\frac{1}{2}} \left(\mathbf{Q}^{-\frac{1}{2}}\mathbf{P}\mathbf{Q}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \mathbf{Q}^{-\frac{1}{2}} \end{aligned} \tag{65}$$

We then have:

$$\begin{aligned} \mathbf{E}^T\mathbf{P}^{-1}\mathbf{E} &= \mathbf{E}^T\mathbf{E}^T\mathbf{P}^{-1} \\ &= (\mathbf{P}\mathbf{Q}^{-1})^T\mathbf{P}^{-1} \\ &= (\mathbf{Q}^{-1}\mathbf{P})\mathbf{P}^{-1} \\ &= \mathbf{Q}^{-1}. \end{aligned} \tag{66}$$

By repeatedly applying the identity $Tr(\mathbf{UV}) = Tr(\mathbf{VU})$ for any $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{n\times n}$, we obtain:

$$\begin{aligned} Tr(\mathbf{A}_1\mathbf{Q}^{-1}) &= Tr(\mathbf{A}_1\mathbf{E}^T\mathbf{P}^{-1}\mathbf{E}) \\ &= Tr(\mathbf{E}\mathbf{A}_1\mathbf{E}^T\mathbf{P}^{-1}) \end{aligned} \tag{67}$$

Similarly, we have:

$$\begin{aligned} Tr(\mathbf{A}_2\mathbf{Q}^{-1}) &= Tr(\mathbf{A}_2\mathbf{E}^T\mathbf{P}^{-1}\mathbf{E}) \\ &= Tr(\mathbf{E}\mathbf{A}_2\mathbf{E}^T\mathbf{P}^{-1}) \end{aligned} \tag{68}$$

From (66), we also get:

$$\begin{aligned} Tr(\mathbf{A}_1\mathbf{Q}^{-1}\mathbf{A}_2\mathbf{Q}^{-1}) &= Tr(\mathbf{A}_1\mathbf{E}^T\mathbf{P}^{-1}\mathbf{E}\mathbf{A}_2\mathbf{E}^T\mathbf{P}^{-1}\mathbf{E}) \\ &= Tr(\mathbf{E}\mathbf{A}_1\mathbf{E}^T\mathbf{P}^{-1}\mathbf{E}\mathbf{A}_2\mathbf{E}^T\mathbf{P}^{-1}). \end{aligned} \tag{69}$$

Combining (67), (68), and (69), we get the conclusion of the Lemma:

$$< \mathbf{A}_1, \mathbf{A}_2 >_{\mathbf{Q}} = < \mathbf{E}\mathbf{A}_1\mathbf{E}^T, \mathbf{E}\mathbf{A}_2\mathbf{E}^T >_{\mathbf{P}} \tag{70}$$

□

4. Proof of Theorem 1

Proof. First, we prove that the group product $\star$ is associative. Let $(\mathbf{P}^m_i, \mathbf{P}^c_i) \in \mathcal{M}(n, n'), i = 1, 2, 3$ and $\mathbf{P}^c_i = \mathbf{L}_i\mathbf{L}_i^T$, where $\mathbf{L}_i$ is a lower triangular matrix with positive diagonal entries. We note that:

$$\begin{aligned} & ((\mathbf{P}^m_1, \mathbf{P}^c_1) \star (\mathbf{P}^m_2, \mathbf{P}^c_2)) \star (\mathbf{P}^m_3, \mathbf{P}^c_3) \\ & = (\varphi^{-1}(\varphi(\mathbf{P}^m_1)(\mathbf{L}_2\mathbf{L}_3) + \varphi(\mathbf{P}^m_2)\mathbf{L}_3 \\ & + \varphi(\mathbf{P}^m_3)), (\mathbf{L}_1\mathbf{L}_2\mathbf{L}_3)(\mathbf{L}_1\mathbf{L}_2\mathbf{L}_3)^T) \\ & = (\mathbf{P}^m_1, \mathbf{P}^c_1) \star ((\mathbf{P}^m_2, \mathbf{P}^c_2) \star (\mathbf{P}^m_3, \mathbf{P}^c_3)). \end{aligned} \tag{71}$$

The neutral element is $(\mathbf{I}_n, \mathbf{I}_{n'})$ since:

$$\begin{aligned} (\mathbf{I}_n, \mathbf{I}_{n'}) \star (\mathbf{P}^m, \mathbf{P}^c) &= (\varphi^{-1}(\varphi(\mathbf{P}^m)), (\mathbf{I}_{n'}\mathbf{L})(\mathbf{I}_{n'}\mathbf{L})^T) \\ &= (\varphi^{-1}(\varphi(\mathbf{P}^m)), \mathbf{L}\mathbf{L}^T) = (\mathbf{P}^m, \mathbf{P}^c), \end{aligned} \tag{72}$$

where $\mathbf{P}^c = \mathbf{L}\mathbf{L}^T$.

Similarly, we have:

$$\begin{aligned} (\mathbf{P}^m, \mathbf{P}^c) \star (\mathbf{I}_n, \mathbf{I}_{n'}) &= (\varphi^{-1}(\varphi(\mathbf{P}^m)\mathbf{I}_{n'}), (\mathbf{L}\mathbf{I}_{n'})(\mathbf{L}\mathbf{I}_{n'})^T) \\ &= (\varphi^{-1}(\varphi(\mathbf{P}^m)), \mathbf{L}\mathbf{L}^T) = (\mathbf{P}^m, \mathbf{P}^c). \end{aligned} \tag{73}$$

Finally, the inverse of $(\mathbf{P}^m, \mathbf{P}^c)$ is given by:

$$(\mathbf{P}^m, \mathbf{P}^c)^{-1} = (\varphi^{-1}(-\varphi(\mathbf{P}^m)\mathbf{L}^{-1}), \mathbf{L}^{-1}\mathbf{L}^{-T}). \tag{74}$$

This can be seen by:

$$
\begin{aligned}
&(\mathbf{P}^m, \mathbf{P}^c) \star (\varphi^{-1}(-\varphi(\mathbf{P}^m)\mathbf{L}^{-1}), \mathbf{L}^{-1}\mathbf{L}^{-T}) \\
&= (\varphi^{-1}(\varphi(\mathbf{P}^m)\mathbf{L}^{-1} - \varphi(\mathbf{P}^m)\mathbf{L}^{-1}), (\mathbf{L}\mathbf{L}^{-1})(\mathbf{L}\mathbf{L}^{-1})^T) \\
&= (\mathbf{I}_n, \mathbf{I}_{n'}).
\end{aligned} \tag{75}
$$

The inverse of $(\mathbf{P}^m, \mathbf{P}^c)$ is unique due to the uniqueness of Cholesky decomposition of SPD matrices and the property of mapping φ. Thus, $\mathcal{M}(n, n')$ is a group. Furthermore, both the group product and the map that sends each element to its inverse are smooth, showing that $\mathcal{M}(n, n')$ is a Lie group. □

5. Proof of Theorem 2

Proof. First, it is easy to see that $K^+(n' + k')$ forms a Lie group since it is a closed subgroup of $GL^+(n' + k')$. Now suppose that $\mathbf{P}_1^c = \mathbf{L}_1\mathbf{L}_1^T$, $\mathbf{P}_2^c = \mathbf{L}_2\mathbf{L}_2^T$ where $\mathbf{L}_1, \mathbf{L}_2 \in LT^+(n')$. Then we have:

$$
\begin{aligned}
\mathbf{K}_{\mathbf{P}_1^m, \mathbf{P}_1^c}\mathbf{K}_{\mathbf{P}_2^m, \mathbf{P}_2^c} &= \begin{bmatrix} \mathbf{L}_1\mathbf{L}_2 & \mathbf{0}_{n' \times k'} \\ \varphi(\mathbf{P}_1^m)\mathbf{L}_2 + \varphi(\mathbf{P}_2^m) & \mathbf{I}_{k'} \end{bmatrix} \\
&= \mathbf{K}_{\varphi^{-1}(\varphi(\mathbf{P}_1^m)\mathbf{L}_2 + \varphi(\mathbf{P}_2^m)), \mathbf{L}_1\mathbf{L}_2}.
\end{aligned} \tag{76}
$$

Therefore:

$$
\phi(\mathbf{K}_{\mathbf{P}_1^m, \mathbf{L}_1}\mathbf{K}_{\mathbf{P}_2^m, \mathbf{L}_2}) = \phi(\mathbf{K}_{\varphi^{-1}(\varphi(\mathbf{P}_1^m)\mathbf{L}_2 + \varphi(\mathbf{P}_2^m)), \mathbf{L}_1\mathbf{L}_2}). \tag{77}
$$

According to the definition of ϕ, the right-hand side of (77) is given by:

$$
(\varphi^{-1}(\varphi(\mathbf{P}_1^m)\mathbf{L}_2 + \varphi(\mathbf{P}_2^m)), (\mathbf{L}_1\mathbf{L}_2)(\mathbf{L}_1\mathbf{L}_2)^T), \tag{78}
$$

which is equal to $(\mathbf{P}_1^m, \mathbf{P}_1^c) \star (\mathbf{P}_2^m, \mathbf{P}_2^c)$ by the definition of the group product $\star$. Thus, we have:

$$
\phi(\mathbf{K}_{\mathbf{P}_1^m, \mathbf{L}_1}\mathbf{A}_{\mathbf{P}_2^m, \mathbf{L}_2}) = (\mathbf{P}_1^m, \mathbf{P}_1^c) \star (\mathbf{P}_2^m, \mathbf{P}_2^c). \tag{79}
$$

Since $(\mathbf{P}_1^m, \mathbf{P}_1^c) = \phi(\mathbf{K}_{\mathbf{P}_1^m, \mathbf{L}_1})$ and $(\mathbf{P}_2^m, \mathbf{P}_2^c) = \phi(\mathbf{K}_{\mathbf{P}_2^m, \mathbf{L}_2})$, we get:

$$
\phi(\mathbf{K}_{\mathbf{P}_1^m, \mathbf{L}_1}\mathbf{K}_{\mathbf{P}_2^m, \mathbf{L}_2}) = \phi(\mathbf{K}_{\mathbf{P}_1^m, \mathbf{L}_1}) \star \phi(\mathbf{K}_{\mathbf{P}_2^m, \mathbf{L}_2}). \tag{80}
$$

We then conclude that mapping ϕ is a group homomorphism. Due to the uniqueness and smoothness of the Cholesky decomposition, ϕ is smooth and bijective and the inverse mapping ϕ^{-1} is smooth. Therefore, ϕ is a Lie group isomorphism. □

Sets	NTU RGB+D 60 (11 classes)		NTU RGB+D 120 (26 classes)	
	X-Subject	X-View	X-Subject	X-Setup
Train	7319	6889	13072	11864
Test	3028	3458	11660	12868

Table 1: The numbers of training and testing samples for NTU RGB+D 60 and NTU RGB+D 120 datasets.

Dataset	SBU	NTU-60		NTU-120	
		X-Subject	X-View	X-Subject	X-Setup
SPDNetBN [4]	86.78	75.24	76.31	61.11	62.36
GeomNet	**96.33**	**93.62**	**96.32**	**86.49**	**87.58**

Table 2: Comparison between GeomNet and SPDNetBN.

6. More details on the datasets and experimental settings

Table 1 gives the numbers of training and testing samples for the experimental protocols on NTU RGB+D 60 and NTU RGB+D 120 datasets. Note that we only used the mutual actions for the experiments on these datasets. All sequences of a dataset were interpolated to have the same number of frames. The number of frames of a sequence in NTU datasets and that of a sequence in SBU Interaction dataset were set to 300 and 45, respectively.

Experimental protocols on NTU RGB+D 60 dataset. For X-subject protocol, training data contains 20 subjects, and testing data contains the other 20 subjects. For X-view protocol, training data comes from the camera views 2 and 3, and testing data comes from the camera view 1.

Experimental protocols on NTU RGB+D 120 dataset. For X-subject protocol, the 106 subjects are split into training and testing groups where each group contains 53 subjects. For X-setup protocol, training data contains samples with even setup IDs, and testing data contains samples with odd setup IDs.

7. More results

Here we compare GeomNet and SPDNetBN [4] using the code[1] published by its authors. SPDNetBN improves SPDNet by introducing a batch normalization layer between Bimap and ReEig layers. This layer is inspired from the classical batch normalization layer in convolutional neural networks and is designed to respect the Riemannian geometry of SPD matrices. Similarly to SPDNet, SPDNetBN only works with the first-order information on SPD manifolds. Results of GeomNet and SPDNetBN[2] are given in Table 2. For SBU dataset, SPDNetBN achieves the best

[1] https://proceedings.neurips.cc/paper/2019/hash/6e69ebbfad976d4637bb4b39de261bf7-Abstract.html

[2] The results are averaged over 3 runs using 500 epochs.

accuracy using our proposed embedding of Gaussians with $k = 1$. For NTU datasets, SPDNetBN achieves the best accuracy using only the covariance information. The results again confirm the superiority of GeomNet over SPD neural networks based only on the first-order information on SPD manifolds.

References

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