1. Extended Quintessential Manifold $\mathcal{Q}(r)$

1.1. Projection on Tangent Space

**Lemma 1.** Projection on the tangent space of the Extended Quintessential manifold is given by:

\[
\tilde{Z} = Z - V\text{sym}(V^T Z) - \frac{V_t \tilde{Z}_t}{1 - \|V_t^T V_t\|^2} (I_{4r} - V V^T)GV
\]

where $G \doteq I_r \otimes (e_1^T e_1)$.  

**Proof.** Since $\mathcal{Q}(r)$ is a Riemannian submanifold, the orthogonal projection of the Euclidean gradient gives its Riemannian gradient. Therefore, we want to find the matrix $\Delta$ such that $\text{Proj}_V(Z) = Z + \Delta$ such that $\|\Delta\|^2_{\tilde{F}}$ is minimal. If we denote the orthogonal complement of $V$ by $V_\perp \in \text{St}(4r - 3, 4r)$, then the $4r \times 4r$ matrix $[V \ V_\perp]$ is in $O(4r)$. Using this, we can write $\Delta$ as $\Delta = VS + VK + V_\perp L$ for some symmetric and skew-symmetric matrices $S$, $K$ and some arbitrary matrix $L$. One can show that $\|\Delta\|^2 = \|S\|^2 + \|K\|^2 + \|L\|^2$.  

First, we need $V^T (Z + \Delta)$ to be skew-symmetric, leading to $\text{sym}(V^T Z + S + K) = 0$, due to $V^T V_\perp = 0$. Also, we have $\text{sym}(S) = S$ and $\text{sym}(K) = 0$, yielding $S = -\text{sym}(V^T Z)$. This gives us $\tilde{Z}$ once we substitute $S$ back in $\Delta$.  

For the second constraint, we want $\langle Z_t + \Delta_t, V_t \rangle$ to be zero, i.e., $\langle G, ZV^T + VKV^T + V_\perp L V^T \rangle = 0$. Since $G$ is symmetric and $VKV^T$ is skew-symmetric, their inner product is zero. After removing $VKV^T$ from the constraint, forming the Lagrangian, and taking derivative with respect to $L$, we get $2L + \lambda V_\perp^T GV = 0$. Placing this expression for $L$ into the constraint and making use of the identity $V_\perp V_\perp^T = I_{4r} - V V^T$ and $G^2 = G$, we find $\lambda$ to be

\[
\lambda = \frac{2\langle G, \tilde{V} \tilde{Z}^T \rangle}{\langle G, V V^T G V_\perp V_\perp^T \rangle} = \frac{2\langle V_t, \tilde{Z}_t \rangle}{1 - \|V_t^T V_t\|^2}
\]

Substituting $L = -\frac{\lambda}{2} V_\perp^T GV$ back in $\Delta$, proof becomes complete. $\square$

1.2. Random Sampling on $\mathcal{Q}(r)$

Here, we present the derivations of the formula for finding $V^*_E$, given by the last two terms of the following equations

\[
V_t^* = \tilde{V}_t \|\tilde{V}_t\|_{\tilde{F}}^{-1}, \\
K = (I_3 - V^T V^*)^{\frac{1}{2}},
\]

(2)

Since a matrix $V^* \in \mathcal{Q}(r) \subset \text{St}(3, 4r)$, it must satisfy $V^* V^* = V^* V^T V^* + V^* V^T V^* = I_3$. Using the definition of $K$, we need to find $V^*_E$ that satisfies $V^*_E V^*_E = K^2$. We intend to find a matrix $X$ closest to $\tilde{V}_E$ that satisfies this condition, which leads to the following Lagrangian function

\[
\mathcal{L}(X, \Lambda) = \|X - \tilde{V}_E\|_{\tilde{F}}^2 + \langle \Lambda, X^T X - K^2 \rangle.
\]

Taking the derivative with respect to $X$, we get $X = V^*_E + X \Lambda = 0$ or $V^*_E = X(I_3 + \Lambda)$. Multiplying both sides with their transpose and denoting the symmetric $I_3 + \Lambda$ by $M$, we get $V^*_E V^*_E = MK^2 M$. If we multiply both sides of this equality by $K$, we get $K V^*_E V^*_E K = (KMK)^2$. Solving for $M$ and using $X = V^*_E M^T$, we get the expression for $V^*_E$ as given in (2).

1.3. Gradient and Hessian

We can find the Riemannian gradient of $h(V)$ by simply projecting the Euclidean gradient on the tangent space of $\mathcal{Q}(r)$

\[
\text{grad} h(V) = \nabla h(V) - V\text{sym}(V^T \nabla h(V)) - c(V, \nabla h(V))(I_{4r} - V V^T)GV
\]

where

\[
c(V, F) \doteq \frac{\langle GV, F - V\text{sym}(V^T F) \rangle}{1 - \|V_t^T V_t\|^2}.
\]

(3)
The projection of the derivative of the gradient vector field gives the Riemannian Hessian. This derivative is given by

\[
D(\nabla^2 h(V))(V) = \nabla^2 h(V) \left[ \frac{\partial}{\partial V} \nabla h(V) \right] - \frac{\partial}{\partial V} \nabla^2 h(V) - c(\cdot)G V - c(\cdot)(G V - \nabla V^T G V). \tag{5}
\]

In the derivations above, we omitted the terms of the form \(\nabla V \cdot Y\) with a symmetric \(S\) as these will be removed by the first step of the projection on the tangent space. Now as for the derivative of \(c(\cdot)\), if we denote its nominator and denominator by \(n_c, d_c\), we have

\[
\frac{\partial}{\partial V} c(V, \nabla h(V)) = \frac{n_c}{d_c} - \frac{n_c d_c}{d_c^2} \tag{6}
\]

and the derivatives of the nominator and denominator are given by

\[
\frac{\partial}{\partial V} n_c = \left( \langle G V, \nabla h(V) - V \text{sym}(\nabla^2 h(V)) \rangle + \langle G V, V^T \nabla^2 h(V) - V \text{sym}(\nabla^2 h(V)) \rangle \right)
\]

\[
- \left( V \text{sym}(\nabla^2 h(V) + \nabla V^T \nabla^2 h(V)) \right) \tag{7}
\]

\[
\frac{\partial}{\partial V} d_c = -2(V^T V_i + V_i^T V_i - V_i^T V_i) = -4(V^T V_i + V_i^T V_i) = -4(V_i^T V_i).
\]

Given these, one can find the Hessian using the projection of \(D(\nabla^2 h(V))\) on the tangent space at \(V\).

1.4. Certificate Matrix

Given a first-order optimal point \(V^* = \ell(Y^*)\) of the rank-restricted problem, it satisfies \(\nabla h(V^*) = 0\) or \(\nabla g(Y^*) = 0\). Using the mapping between the two variable arrangements, we can find \(\nabla g(Y)\) using the projection given in (1) and the expression given in (3)

\[
\nabla g(Y) = \ell^{-1}(\text{Proj}_{\ell}(Y))(\ell(2 CY)) = \text{Proj}_{\ell}(2 CY).
\]

The projection for the \(Y\) arrangement is thus given by

\[
\hat{Z} = Z - (M \otimes I_4) Y
\]

\[
\text{Proj}_Y(Z) = \hat{Z} - (I_3 \otimes (e_4 e_4^T) - N \otimes I_4) c(\hat{Z}) Y \tag{9}
\]

where \(M, N \in \mathbb{R}^{3 \times 3}\) are such that \(m_{ij} = \frac{1}{2}((Y_i, Z_j) + (Y_j, Z_i))\) and \(n_{ij} = \langle e_4 e_4^T, Y_i^T Y_j \rangle\), and the function \(c(\cdot)\) is given by (4). Once we project \(2 CY\) onto the tangent space, we get the Riemannian gradient. Given the structure of \(C\), for \(Z = 2 CY\) we have \(c(\hat{Z}) = 0\) due to \(Z_i, Z_i\) being equal to zero. Therefore, the projection of the gradient is obtained by the first step of the projection process, yielding

\[
\nabla g(Y^*) = 0 \rightarrow (C - (M \otimes I_4)) Y^* = 0. \tag{10}
\]

This gives us \(S = C - (M \otimes I_4)\) satisfying \(SY^* = 0\).

2. Local Solver

2.1. Projection on Tangent Space

Lemma 2. Projection on the tangent space of \((QO)^2\) is given by:

\[
\hat{Z}_i = Z_i - O_i \text{sym}(O_i^T Z_i) \tag{11}
\]

\[
\text{Proj}_{O_i}(Z_i) = \hat{Z}_i - \frac{\sum_{i \neq j} (O_i^T \hat{Z}_j)}{2} O_i \text{skew}(O_i^T O_j F) \]

\[
\sum_{i \neq j} (F, O_i^T \hat{Z}_j + O_i^T O_j K_j) = 0. \tag{12}
\]

Since this is the sum of two inner products between the last columns of \(O_i, Z_j, O_i K_i\), the matrices \(K_1, K_2\) need only have non-zero entries in their fourth rows and columns. Noting that the right bottom entry of \(K_1, K_2\) is zero, we denote their fourth column minus the last (zero) entry as \(k_1, k_2 \in \mathbb{R}^3\). If we denote the epipole appearing in the bottom row of \(Q = O_i^T O_j\) by \(t_i\) and the other epipole by \(t_j\), we can rewrite (12) as \(t_i^T k_2 + t_j^T k_1 = -\sum_{i \neq j} (F, O_i^T Z_j)\).

Due to \(\|k_i\|^2 + \|k_j\|^2 + \lambda(t_i^T k_2 + t_j^T k_1 + \sum_{i \neq j} (F, O_i^T Z_j))\) and find the optimal \(k_1, k_2\), which concludes the proof. \(\square\)

2.2. Gradient and Hessian

For the algebraic error, we have \(f(Q)\) as

\[
f(Q) = \sum_{k=1}^{N} (Q, \tilde{f}_{i,k} \tilde{f}_{j,k}^T)^2, \tag{13}
\]

and its gradient and hessian are given as

\[
\nabla Q f(Q) = 2 \sum_{k=1}^{N} (Q, \tilde{f}_{i,k} \tilde{f}_{j,k}^T) \tilde{f}_{i,k} \tilde{f}_{j,k}^T, \tag{14}
\]

\[
\nabla^2 Q f(Q)(\tilde{Q}) = 2 \sum_{k=1}^{N} (Q, \tilde{f}_{i,k} \tilde{f}_{j,k}^T) \tilde{f}_{i,k} \tilde{f}_{j,k}^T, \tag{15}
\]

where \(\tilde{f}_{i,k} = [\tilde{f}_{i,k}^T, 0]^T\) and \(\tilde{f}_{i,k} = [\tilde{f}_{i,k}^T, 0]^T\).

Taking gradient of \(f(Q) = f(O_1^T O_2)\) with respect to \(O_1\) and \(O_2\) gives

\[
\nabla_{O_1} f(Q) = 2 O_2 \nabla Q f(Q)^T, \tag{15}
\]

\[
\nabla_{O_2} f(Q) = 2 O_1 \nabla Q f(Q), \tag{15}
\]
and their Hessian is given by
\[
\nabla^2_{\dot{O}_1} f(Q)[\dot{O}_1] = 2O_2 \nabla^2_Q f(Q)[\dot{O}_1^T O_2]^T,
\]

\[
\nabla^2_{\dot{O}_2} f(Q)[\dot{O}_2] = 2O_1 \nabla^2_Q f(Q)[O_1^T \dot{O}_2].
\]

(16)

As before, the Riemannian Hessian is given by projecting the differential of the gradient. The gradient and its differential are given as
\[
\text{grad}_{\dot{O}_i} f(Q) = \nabla f(Q) - O_i \text{sym}(O_i^T \nabla f(Q))
- c_i(\cdot)O_i \text{skew}(O_i^T O_j F),
\]

\[
D(\text{grad}_{\dot{O}_i} f(Q)) = \nabla^2_{\dot{O}_i} f(Q)[\dot{O}_i]
- \dot{O}_i \text{sym}(O_i^T \nabla f(Q))
- \dot{c}_i(\cdot)O_i \text{skew}(O_i^T O_j F)
- c_i(\cdot)\dot{O}_i \text{skew}(O_i^T O_j F)
- c_i(\cdot)\dot{O}_i \text{skew}(O_i^T O_j F),
\]

where the function \(c_i(\cdot)\) and its differential are given by
\[
c_i(O_{1,2}, \nabla_{O_{1,2}} f(Q)) =
\frac{1}{2} \sum_{i \neq j} \langle O_i^T (\nabla_{O_j} f(Q) - O_j \text{sym}(O_j^T \nabla_{O_j} f(Q))), F \rangle,
\]

\[
\dot{c}_i(O_{1,2}, \nabla_{O_{1,2}} f(Q)) =
\frac{1}{2} \langle \dot{O}_i^T (\nabla_{O_j} f(Q) - O_j \text{sym}(O_j^T \nabla_{O_j} f(Q))), F \rangle
+ \frac{1}{2} \langle O_j^T (\nabla^2_{\dot{O}_i} f(Q)[\dot{O}_i] - \dot{O}_i \text{sym}(O_i^T \nabla_{O_i} f(Q))
- O_i \text{sym}(O_i^T \nabla f(Q) + O_i^T \nabla^2_{\dot{O}_i} f(Q)[\dot{O}_i]), F \rangle.
\]

(17)