Essential Matrix Estimation using Convex Relaxations in Orthogonal Space Supplementary Material

Arman Karimian
Boston University
armandok@bu.edu

Roberto Tron
Boston University
tron@bu.edu

1. Extended Quintessential Manifold $\mathfrak{Q}(r)$

1.1. Projection on Tangent Space

Lemma 1. *Projection on the tangent space of the Extended Quintessential manifold is given by:*

$$\begin{aligned}\tilde{\mathbf{Z}} &= \mathbf{Z} - \mathbf{V}\mathrm{sym}(\mathbf{V}^\top\mathbf{Z}) \\ \mathrm{Proj}_{\mathbf{V}}(\mathbf{Z}) &= \tilde{\mathbf{Z}} - \frac{\langle \mathbf{V}_t, \tilde{\mathbf{Z}}_t\rangle}{1 - \|\mathbf{V}_t^\top\mathbf{V}_t\|_{\mathrm{F}}^2}(\mathbf{I}_{4r} - \mathbf{V}\mathbf{V}^\top)\mathbf{G}\mathbf{V}\end{aligned} \tag{1}$$

where $\mathbf{G} \doteq \mathbf{I}_r \otimes (\mathbf{e}_4\mathbf{e}_4^\top)$.

Proof. Since $\mathfrak{Q}(r)$ is a Riemannian submanifold, the orthogonal projection of the Euclidean gradient gives its Riemannian gradient. Therefore, we want to find the matrix $\mathbf{\Delta}$ such that $\mathrm{Proj}_{\mathbf{V}}(\mathbf{Z}) = \mathbf{Z} + \mathbf{\Delta}$ such that $\|\mathbf{\Delta}\|_{\mathrm{F}}^2$ is minimal. If we denote the orthogonal complement of $\mathbf{V}$ by $\mathbf{V}_\perp \in \mathrm{St}(4r-3, 4r)$, then the $4r \times 4r$ matrix $[\mathbf{V} \;\; \mathbf{V}_\perp]$ is in $\mathrm{O}(4r)$. Using this, we can write $\mathbf{\Delta}$ as $\mathbf{\Delta} = \mathbf{V}\mathbf{S} + \mathbf{V}\mathbf{K} + \mathbf{V}_\perp\mathbf{L}$ for some symmetric and skew-symmetric matrices $\mathbf{S}, \mathbf{K}$ and some arbitrary matrix $\mathbf{L}$. One can show that $\|\mathbf{\Delta}\|_{\mathrm{F}}^2 = \|\mathbf{S}\|_{\mathrm{F}}^2 + \|\mathbf{K}\|_{\mathrm{F}}^2 + \|\mathbf{L}\|_{\mathrm{F}}^2$.
First, we need $\mathbf{V}^\top(\mathbf{Z}+\mathbf{\Delta})$ to be skew-symmetric, leading to $\mathrm{sym}(\mathbf{V}^\top\mathbf{Z}+\mathbf{S}+\mathbf{K}) = \mathbf{0}$, due to $\mathbf{V}^\top\mathbf{V}_\perp = \mathbf{0}$. Also, we have $\mathrm{sym}(\mathbf{S}) = \mathbf{S}$ and $\mathrm{sym}(\mathbf{K}) = \mathbf{0}$, yielding $\mathbf{S} = -\mathrm{sym}(\mathbf{V}^\top\mathbf{Z})$. This gives us $\tilde{\mathbf{Z}}$ once we substitute $\mathbf{S}$ back in $\mathbf{\Delta}$.
For the second constraint, we want $\langle \mathbf{Z}_t + \mathbf{\Delta}_t, \mathbf{V}_t\rangle$ to be zero, i.e., $\langle \mathbf{G}, \tilde{\mathbf{Z}}\mathbf{V}^\top + \mathbf{V}\mathbf{K}\mathbf{V}^\top + \mathbf{V}_\perp\mathbf{L}\mathbf{V}^\top\rangle = 0$. Since $\mathbf{G}$ is symmetric and $\mathbf{V}\mathbf{K}\mathbf{V}^\top$ is skew-symmetric, their inner product is zero. After removing $\mathbf{V}\mathbf{K}\mathbf{V}^\top$ from the constraint, forming the Lagrangian, and taking derivative with respect to $\mathbf{L}$, we get $2\mathbf{L} + \lambda\mathbf{V}_\perp^\top\mathbf{G}\mathbf{V} = \mathbf{0}$. Placing this expression for $\mathbf{L}$ into the constraint and making use of the identity $\mathbf{V}_\perp\mathbf{V}_\perp^\top = \mathbf{I}_{4r} - \mathbf{V}\mathbf{V}^\top$ and $\mathbf{G}^2 = \mathbf{G}$, we find λ to be

$$\lambda = \frac{2\langle \mathbf{G}, \mathbf{V}\tilde{\mathbf{Z}}^\top\rangle}{\langle \mathbf{G}, \mathbf{V}\mathbf{V}^\top\mathbf{G}\mathbf{V}_\perp\mathbf{V}_\perp^\top\rangle} = \frac{2\langle \mathbf{V}_t, \tilde{\mathbf{Z}}_t\rangle}{1 - \|\mathbf{V}_t^\top\mathbf{V}_t\|_{\mathrm{F}}^2}$$

Substituting $\mathbf{L} = -\frac{\lambda}{2}\mathbf{V}_\perp^\top\mathbf{G}\mathbf{V}$ back in $\mathbf{\Delta}$, proof becomes complete. □

1.2. Random Sampling on $\mathfrak{Q}(r)$

Here, we present the derivations of the formula for finding $\mathbf{V}_E^\bullet$, given by the last two terms of the following equations

$$\begin{aligned}\mathbf{V}_t^\bullet &= \tilde{\mathbf{V}}_t\|\tilde{\mathbf{V}}_t\|_{\mathrm{F}}^{-1}, \\ \mathbf{K} &= (\mathbf{I}_3 - \mathbf{V}_t^{\bullet\top}\mathbf{V}_t^\bullet)^{\frac{1}{2}}, \\ \mathbf{V}_E^\bullet &= \tilde{\mathbf{V}}_E\mathbf{K}(\mathbf{K}\tilde{\mathbf{V}}_E^\top\tilde{\mathbf{V}}_E\mathbf{K})^{\frac{\dagger}{2}}\mathbf{K}.\end{aligned} \tag{2}$$

Since a matrix $\mathbf{V}^\bullet \in \mathfrak{Q}(r) \subset \mathrm{St}(3, 4r)$, it must satisfy $\mathbf{V}^{\bullet\top}\mathbf{V}^\bullet = \mathbf{V}_t^{\bullet\top}\mathbf{V}_t^\bullet + \mathbf{V}_E^{\bullet\top}\mathbf{V}_E^\bullet = \mathbf{I}_3$. Using the definition of $\mathbf{K}$, we need to find $\mathbf{V}_E^\bullet$ that satisfies $\mathbf{V}_E^{\bullet\top}\mathbf{V}_E^\bullet = \mathbf{K}^2$. We intend to find a matrix $\mathbf{X}$ closest to $\tilde{\mathbf{V}}_E$ that satisfies this condition, which leads to the following Lagrangian function

$$\mathcal{L}(\mathbf{X}, \mathbf{\Lambda}) = \|\mathbf{X} - \tilde{\mathbf{V}}_E\|_{\mathrm{F}}^2 + \langle \mathbf{\Lambda}, \mathbf{X}^\top\mathbf{X} - \mathbf{K}^2\rangle.$$

Taking the derivative with respect to $\mathbf{X}$, we get $\mathbf{X} - \tilde{\mathbf{V}}_E + \mathbf{X}\mathbf{\Lambda} = \mathbf{0}$ or $\tilde{\mathbf{V}}_E = \mathbf{X}(\mathbf{I}_3 + \mathbf{\Lambda})$. Multiplying both sides with their transpose and denoting the symmetric $\mathbf{I}_3 + \mathbf{\Lambda}$ by $\mathbf{M}$, we get $\tilde{\mathbf{V}}_E^\top\tilde{\mathbf{V}}_E = \mathbf{M}\mathbf{K}^2\mathbf{M}$. If we multiply both sides of this equality by $\mathbf{K}$, we get $\mathbf{K}\tilde{\mathbf{V}}_E^\top\tilde{\mathbf{V}}_E\mathbf{K} = (\mathbf{K}\mathbf{M}\mathbf{K})^2$. Solving for $\mathbf{M}$ and using $\mathbf{X} = \tilde{\mathbf{V}}_E\mathbf{M}^\dagger$, we get the expression for $\mathbf{V}_E^\bullet$ as given in (2).

1.3. Gradient and Hessian

We can find the Riemannian gradient of $h(\mathbf{V})$ by simply projecting the Euclidean gradient on the tangent space of $\mathfrak{Q}(r)$

$$\begin{aligned}\mathrm{grad}\, h(\mathbf{V}) &= \nabla h(\mathbf{V}) - \mathbf{V}\mathrm{sym}(\mathbf{V}^\top\nabla h(\mathbf{V})) \\ &\quad - c(\mathbf{V}, \nabla h(\mathbf{V}))(\mathbf{I}_{4r} - \mathbf{V}\mathbf{V}^\top)\mathbf{G}\mathbf{V}\end{aligned} \tag{3}$$

where

$$c(\mathbf{V}, \mathbf{F}) \doteq \frac{\langle \mathbf{G}\mathbf{V}, \mathbf{F} - \mathbf{V}\mathrm{sym}(\mathbf{V}^\top\mathbf{F})\rangle}{1 - \|\mathbf{V}_t^\top\mathbf{V}_t\|_{\mathrm{F}}^2}. \tag{4}$$

The projection of the derivative of the gradient vector field gives the Riemannian Hessian. This derivative is given by

$$\begin{aligned} \mathrm{D}\big(\operatorname{grad} h(\mathbf{V})\big)(\mathbf{V})[\dot{\mathbf{V}}] &= \nabla^2 h(\mathbf{V})[\dot{\mathbf{V}}] \\ &- \dot{\mathbf{V}}\operatorname{sym}(\mathbf{V}^\mathsf{T}\nabla h(\mathbf{V})) - \dot{c}(\cdot)\mathbf{G}\mathbf{V} \\ &- c(\cdot)\big(\mathbf{G}\dot{\mathbf{V}} - \dot{\mathbf{V}}\mathbf{V}^\mathsf{T}\mathbf{G}\mathbf{V}\big). \end{aligned} \tag{5}$$

In the derivations above, we omitted the terms of the form $\mathbf{VS}$ with a symmetric $\mathbf{S}$ as these will be removed by the first step of the projection on the tangent space. Now as for the derivative of $c(\cdot)$, if we denote its nominator and denominator by n_c, d_c, we have

$$\dot{c}(\mathbf{V}, \nabla h(\mathbf{V})) = \frac{\dot{n}_c}{d_c} - \frac{n_c \dot{d}_c}{d_c^2} \tag{6}$$

and the derivatives of the nominator and denominator are given by

$$\begin{aligned} \dot{n}_c &= \Big(\langle \mathbf{G}\dot{\mathbf{V}}, \nabla h(\mathbf{V}) - \mathbf{V}\operatorname{sym}(\mathbf{V}^\mathsf{T}\nabla h(\mathbf{V}))\rangle \\ &+ \langle \mathbf{G}\mathbf{V}, \nabla^2 h(\mathbf{V})[\dot{\mathbf{V}}] - \dot{\mathbf{V}}\operatorname{sym}(\mathbf{V}^\mathsf{T}\nabla h(\mathbf{V})) \\ &- \mathbf{V}\operatorname{sym}(\dot{\mathbf{V}}^\mathsf{T}\nabla h(\mathbf{V}) + \mathbf{V}^\mathsf{T}\nabla^2 h(\mathbf{V})[\dot{\mathbf{V}}])\rangle\Big) \\ \dot{d}_c &= -2\langle \dot{\mathbf{V}}_t^\mathsf{T}\mathbf{V}_t + \mathbf{V}_t^\mathsf{T}\dot{\mathbf{V}}_t, \mathbf{V}_t^\mathsf{T}\mathbf{V}_t\rangle \\ &= -4\langle \dot{\mathbf{V}}_t^\mathsf{T}\mathbf{V}_t, \mathbf{V}_t^\mathsf{T}\mathbf{V}_t\rangle \end{aligned} \tag{7}$$

Given these, one can find the Hessian using the projection of $\mathrm{D}\big(\operatorname{grad} h(\mathbf{V})\big)$ on the tangent space at $\mathbf{V}$.

1.4. Certificate Matrix

Given a first-order optimal point $\mathbf{V}^* = \ell(\mathbf{Y}^*)$ of the rank-restricted problem, it satisfies $\operatorname{grad} h(\mathbf{V}^*) = \mathbf{0}$ or $\operatorname{grad} g(\mathbf{Y}^*) = \mathbf{0}$. Using the mapping between the two variable arrangements, we can find $\operatorname{grad} g(\mathbf{Y})$ using the projection given in (1) and the expression given in (3)

$$\operatorname{grad} g(\mathbf{Y}) = \ell^{-1}\big(\mathrm{Proj}_{\ell(\mathbf{Y})}(\ell(2\mathbf{CY}))\big) = \mathrm{Proj}_{\mathbf{Y}}(2\mathbf{CY}). \tag{8}$$

The projection for the $\mathbf{Y}$ arrangement is thus given by

$$\begin{aligned} \tilde{\mathbf{Z}} &= \mathbf{Z} - (\mathbf{M}\otimes\mathbf{I}_4)\mathbf{Y} \\ \mathrm{Proj}_{\mathbf{Y}}(\mathbf{Z}) &= \tilde{\mathbf{Z}} - (\mathbf{I}_3\otimes(\mathbf{e}_4\mathbf{e}_4^\mathsf{T}) - \mathbf{N}\otimes\mathbf{I}_4)c(\tilde{\mathbf{Z}})\mathbf{Y} \end{aligned} \tag{9}$$

where $\mathbf{M}, \mathbf{N} \in \mathbb{R}^{3\times3}$ are such that $m_{ij} = \frac{1}{2}(\langle \mathbf{Y}_i, \mathbf{Z}_j\rangle + \langle \mathbf{Y}_j, \mathbf{Z}_i\rangle)$ and $n_{ij} = \langle \mathbf{e}_4\mathbf{e}_4^\mathsf{T}, \mathbf{Y}_i^\mathsf{T}\mathbf{Y}_j\rangle$, and the function $c(\cdot)$ is given by (4). Once we project $2\mathbf{CY}$ onto the tangent space, we get the Riemannian gradient. Given the structure of $\mathbf{C}$, for $\mathbf{Z} = 2\mathbf{CY}$ we have $c(\tilde{\mathbf{Z}}) = 0$ due to $\mathbf{Z}_t, \tilde{\mathbf{Z}}_t$ being equal to zero. Therefore, the projection of the gradient is obtained by the first step of the projection process, yielding

$$\operatorname{grad} g(\mathbf{Y}^*) = \mathbf{0} \rightarrow \big(\mathbf{C} - (\mathbf{M}\otimes\mathbf{I}_4)\big)\mathbf{Y}^* = \mathbf{0}. \tag{10}$$

This gives us $\mathbf{S} \doteq \mathbf{C} - (\mathbf{M}\otimes\mathbf{I}_4)$ satisfying $\mathbf{SY}^* = \mathbf{0}$.

2. Local Solver

2.1. Projection on Tangent Space

Lemma 2. *Projection on the tangent space of* QO^2 *is given by:*

$$\begin{aligned} \tilde{\mathbf{Z}}_i &= \mathbf{Z}_i - \mathbf{O}_i\operatorname{sym}(\mathbf{O}_i^\mathsf{T}\mathbf{Z}_i) \\ \mathrm{Proj}_{\mathbf{O}_i}(\mathbf{Z}_i) &= \tilde{\mathbf{Z}}_i - \frac{\sum\langle \mathbf{O}_i^\mathsf{T}\tilde{\mathbf{Z}}_j, \mathbf{F}\rangle}{2}\mathbf{O}_i\operatorname{skew}(\mathbf{O}_i^\mathsf{T}\mathbf{O}_j\mathbf{F}) \end{aligned} \tag{11}$$

Proof. Similar to the proof of Lemma 1, we want to find the matrices $\mathbf{\Delta}_i$ such that $\mathrm{Proj}_{\mathbf{O}_i}(\mathbf{Z}_i) = \mathbf{Z}_i + \mathbf{\Delta}_i$ so that $\|\mathbf{\Delta}_1\|_\mathrm{F}^2 + \|\mathbf{\Delta}_2\|_\mathrm{F}^2$ is minimal. Since $\mathbf{O}_i$ is a basis, we can write $\mathbf{\Delta}_i$ as $\mathbf{\Delta}_i = \mathbf{O}_i(\mathbf{S}_i + \mathbf{K}_i)$ for some symmetric and skew-symmetric matrices $\mathbf{S}_i, \mathbf{K}_i$. As $\mathrm{Proj}_{\mathbf{O}_i}(\mathbf{Z}_i)^\mathsf{T}\mathbf{O}_i$ should be skew-symmetric, the symmetric part is given by $\mathbf{S}_i = -\operatorname{sym}(\mathbf{O}_i^\mathsf{T}\mathbf{Z}_i)$. This leaves us $\mathrm{Proj}_{\mathbf{O}_i}(\mathbf{Z}_i) = \tilde{\mathbf{Z}}_i + \mathbf{O}_i\mathbf{K}_i$, and we want $\mathbf{K}_1, \mathbf{K}_2$ to satisfy

$$\sum_{i\neq j}\langle \mathbf{F}, \mathbf{O}_i^\mathsf{T}\tilde{\mathbf{Z}}_j + \mathbf{O}_i^\mathsf{T}\mathbf{O}_j\mathbf{K}_j\rangle = 0. \tag{12}$$

Since this is the sum of two inner products between the last columns of $\mathbf{O}_i, \tilde{\mathbf{Z}}_j + \mathbf{O}_j\mathbf{K}_j$, the matrices $\mathbf{K}_1, \mathbf{K}_2$ need only have non-zero entries in their fourth rows and columns. Noting that the right bottom entry of $\mathbf{K}_1, \mathbf{K}_2$ is zero, we denote their fourth column minus the last (zero) entry as $\mathbf{k}_1, \mathbf{k}_2 \in \mathbb{R}^3$. If we denote the epipole appearing in the bottom row of $\mathbf{Q} = \mathbf{O}_1^\mathsf{T}\mathbf{O}_2$ by $\mathbf{t}_l$ and the other epipole by $\mathbf{t}_r$, we can rewrite (12) as $\mathbf{t}_l^\mathsf{T}\mathbf{k}_2 + \mathbf{t}_r^\mathsf{T}\mathbf{k}_1 = -\sum_{i\neq j}\langle \mathbf{F}, \mathbf{O}_i^\mathsf{T}\tilde{\mathbf{Z}}_j\rangle$. Due to $\|\mathbf{K}_i\|_\mathrm{F}^2 = 2\|\mathbf{k}_i\|^2$, we can use the Lagrangian function $\|\mathbf{k}_1\|^2 + \|\mathbf{k}_2\|^2 + \lambda(\mathbf{t}_l^\mathsf{T}\mathbf{k}_2 + \mathbf{t}_r^\mathsf{T}\mathbf{k}_1 + \sum_{i\neq j}\langle \mathbf{F}, \mathbf{O}_i^\mathsf{T}\tilde{\mathbf{Z}}_j\rangle)$ and find the optimal $\mathbf{k}_1, \mathbf{k}_2$, which concludes the proof. □

2.2. Gradient and Hessian

For the algebraic error, we have $f(\mathbf{Q})$ as

$$f(\mathbf{Q}) = \sum_{k=1}^{N}\langle \mathbf{Q}, \breve{\mathbf{f}}_{i,k}\breve{\mathbf{f}}_{j,k}^\mathsf{T}\rangle^2, \tag{13}$$

and its gradient and hessian are given as

$$\begin{aligned} \nabla_{\mathbf{Q}} f(\mathbf{Q}) &= 2\sum_{k=1}^{N}\langle \mathbf{Q}, \breve{\mathbf{f}}_{i,k}\breve{\mathbf{f}}_{j,k}^\mathsf{T}\rangle\, \breve{\mathbf{f}}_{i,k}\breve{\mathbf{f}}_{j,k}^\mathsf{T}, \\ \nabla_{\mathbf{Q}}^2 f(\mathbf{Q})[\dot{\mathbf{Q}}] &= 2\sum_{k=1}^{N}\langle \dot{\mathbf{Q}}, \breve{\mathbf{f}}_{i,k}\breve{\mathbf{f}}_{j,k}^\mathsf{T}\rangle\, \breve{\mathbf{f}}_{i,k}\breve{\mathbf{f}}_{j,k}^\mathsf{T}, \end{aligned} \tag{14}$$

where $\breve{\mathbf{f}}_{i,k} = [\mathbf{f}_{i,k}^\mathsf{T} 0]^\mathsf{T}$ and $\breve{\mathbf{f}}_{j,k} = [\mathbf{f}_{j,k}^\mathsf{T} 0]^\mathsf{T}$.

Taking gradient of $f(\mathbf{Q}) = f(\mathbf{O}_1^\mathsf{T}\mathbf{O}_2)$ with respect to $\mathbf{O}_1$ and $\mathbf{O}_2$ gives

$$\begin{aligned} \nabla_{\mathbf{O}_1} f(\mathbf{Q}) &= 2\mathbf{O}_2\nabla_{\mathbf{Q}} f(\mathbf{Q})^\mathsf{T}, \\ \nabla_{\mathbf{O}_2} f(\mathbf{Q}) &= 2\mathbf{O}_1\nabla_{\mathbf{Q}} f(\mathbf{Q}), \end{aligned} \tag{15}$$

and their Hessian is given by

$$\begin{aligned}\nabla^2_{\mathbf{O}_1} f(\mathbf{Q})[\dot{\mathbf{O}}_1] &= 2\mathbf{O}_2 \nabla^2_{\mathbf{Q}} f(\mathbf{Q})[\dot{\mathbf{O}}_1^\mathsf{T} \mathbf{O}_2]^\mathsf{T}, \\ \nabla^2_{\mathbf{O}_2} f(\mathbf{Q})[\dot{\mathbf{O}}_2] &= 2\mathbf{O}_1 \nabla^2_{\mathbf{Q}} f(\mathbf{Q})[\mathbf{O}_1^\mathsf{T} \dot{\mathbf{O}}_2].\end{aligned} \tag{16}$$

As before, the Riemannian Hessian is given by projecting the differential of the gradient. The gradient and its differential are given as

$$\begin{aligned}\operatorname{grad}_{\mathbf{O}_i} f(\mathbf{Q})) &= \nabla_{\mathbf{O}_i} f(\mathbf{Q}) - \mathbf{O}_i \operatorname{sym}(\mathbf{O}_i^\mathsf{T} \nabla_{\mathbf{O}_i} f(\mathbf{Q})) \\ &\quad - c_i(\cdot)\mathbf{O}_i \operatorname{skew}(\mathbf{O}_i^\mathsf{T} \mathbf{O}_j \mathbf{F}), \\ \mathrm{D}(\operatorname{grad}_{\mathbf{O}_i} f(\mathbf{Q})) &= \nabla^2_{\mathbf{O}_i} f(\mathbf{Q})[\dot{\mathbf{O}}_i] \\ &\quad - \dot{\mathbf{O}}_i \operatorname{sym}(\mathbf{O}_i^\mathsf{T} \nabla_{\mathbf{O}_i} f(\mathbf{Q})) \\ &\quad - \dot{c}_i(\cdot)\mathbf{O}_i \operatorname{skew}(\mathbf{O}_i^\mathsf{T} \mathbf{O}_j \mathbf{F}) \\ &\quad - c_i(\cdot)\dot{\mathbf{O}}_i \operatorname{skew}(\mathbf{O}_i^\mathsf{T} \mathbf{O}_j \mathbf{F}) \\ &\quad - c_i(\cdot)\mathbf{O}_i \operatorname{skew}(\dot{\mathbf{O}}_i^\mathsf{T} \mathbf{O}_j \mathbf{F}),\end{aligned} \tag{17}$$

where the function $c_i(\cdot)$ and its differential are given by

$$\begin{aligned}&c_i(\mathbf{O}_{1,2}, \nabla_{\mathbf{O}_{1,2}} f(\mathbf{Q})) = \\ &\quad \frac{1}{2}\sum_{i \neq j} \langle \mathbf{O}_i^\mathsf{T} \big(\nabla_{\mathbf{O}_j} f(\mathbf{Q}) - \mathbf{O}_j \operatorname{sym}(\mathbf{O}_j^\mathsf{T} \nabla_{\mathbf{O}_j} f(\mathbf{Q}))\big), \mathbf{F}\rangle, \\ &\dot{c}_i(\mathbf{O}_{1,2}, \nabla_{\mathbf{O}_{1,2}} f(\mathbf{Q})) = \\ &\quad \frac{1}{2}\langle \dot{\mathbf{O}}_i^\mathsf{T} \big(\nabla_{\mathbf{O}_j} f(\mathbf{Q}) - \mathbf{O}_j \operatorname{sym}(\mathbf{O}_j^\mathsf{T} \nabla_{\mathbf{O}_j} f(\mathbf{Q}))\big), \mathbf{F}\rangle \\ &+ \frac{1}{2}\langle \mathbf{O}_j^\mathsf{T} \Big(\nabla^2_{\mathbf{O}_i} f(\mathbf{Q})[\dot{\mathbf{O}}_i] - \dot{\mathbf{O}}_i \operatorname{sym}(\mathbf{O}_i^\mathsf{T} \nabla_{\mathbf{O}_i} f(\mathbf{Q})) \\ &\quad - \mathbf{O}_i \operatorname{sym}(\dot{\mathbf{O}}_i^\mathsf{T} \nabla_{\mathbf{O}_i} f(\mathbf{Q}) + \mathbf{O}_i^\mathsf{T} \nabla^2_{\mathbf{O}_i} f(\mathbf{Q})[\dot{\mathbf{O}}_i])\Big), \mathbf{F}\rangle.\end{aligned} \tag{18}$$