TiDy-PSFs: Computational Imaging with Time-Averaged Dynamic Point-Spread-Functions
Supplementary Material

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Abstract

This supplement includes a more general proof that the set of PSFs described by a single phase mask is non-convex. It also includes an extended discussion of the benefits a single-shot time-averaged systems has over a multi-shot burst imaging system.

1. Generalized Proof of PSF Non-convexity

This proof, similar to the one included in the main paper, will simplify the convexity of the PSF set to the convexity of cross-correlation. We generalize the result for any aperture by showing there always exists a shift such that the overlap of any set of points and their shifts is a single element.

Definition 1. *Let $\mathcal{D} = \{v \in R^2 : \|v\| = 1\}$ be the set of all unit vector directions.*

Definition 2. *Let* setmax *and* setmin *be defined by,*

$$\text{setmax}(\mathcal{S}, v) = \{x \in \mathcal{S} : x \cdot v = \max_{x \in \mathcal{S}} x \cdot v\} \tag{1}$$

$$\text{setmin}(\mathcal{S}, v) = \{x \in \mathcal{S} : x \cdot v = \min_{x \in \mathcal{S}} x \cdot v\}. \tag{2}$$

setmax *produces the set of all points in $\mathcal{S}$ that are furthest in direction v, and* setmin *similarly produces the set of all points that are furthest in the opposite direction of v.*

Lemma 1. *For all finite non-empty sets of points $\mathcal{S}$, there exists some shift δ such that* $\text{card}(\mathcal{S} \cap (\mathcal{S} + \delta)) = 1$ *where $\mathcal{S}+\delta = \{x+\delta : x \in \mathcal{S}\}$, and* $\text{card}(\cdot)$ *denotes the cardinally of a set. That is, there exists some shift such that $\mathcal{S}$ and $\mathcal{S}$ shifted overlap at exactly one point.*

Proof. Consider the set of all directions without a unique maximizer,

$$\mathcal{V} = \{v \in \mathcal{D} : \text{card}(\text{setmax}(\mathcal{S}, v)) > 1\}. \tag{3}$$

Notice that for all $v \in \mathcal{V}$, we can treat v as a normal vector to the line formed by points in $\text{setmax}(\mathcal{S}, v)$ (Figure 1). $\mathcal{V}$ is the set of normal vectors whose corresponding line intersects multiple points of $\mathcal{S}$. We can upper bound $\text{card}(\mathcal{V})$ as the number of unique lines that intersect two points in $\mathcal{S}$.

$$\text{card}(\mathcal{V}) \leq \text{card}(\{\overrightarrow{xy} : x, y \in S\}) < \infty \tag{4}$$

Therefore, $\mathcal{V}$ is a finite set (whereas $\mathcal{D}$, the set of all unit vectors, is clearly an infinite set). Then, there always exists some u such that $u \in \mathcal{D}$ and $u \notin \mathcal{V}$. Because $u \notin \mathcal{V}$, $\text{card}(\text{setmax}(\mathcal{S}, u)) = 1$, the direction u has a unique maximizer. Let m be the single element of $\text{setmax}(\mathcal{S}, u)$, and choose $\delta \in (m - \text{setmin}(\mathcal{S}, u))$. δ is the difference between u's unique maximizer and one of u's minimizers. Observe that $\text{setmax}(\mathcal{S}, u)$ and $\text{setmin}(\mathcal{S}, u)$ define the extents of $\mathcal{S}$ in the direction u (Figure 2). Therefore, when applying the shift δ, only the furthest point in $\mathcal{S}$ in direction u and $-u$ will overlap (Figure 3). Let $\mathcal{T}$ include all points from $\mathcal{S}$ except m. Then, $\mathcal{T}$ and $\mathcal{T}+\delta$ are disjoint by definition. Therefore, $\mathcal{S} \cap (\mathcal{S} + \delta) = \{m\}$, which is a single element. □

The following is similar to the proof included in the main paper; however, we relax the condition on A to be any arbitrary aperture. Therefore, this proof of PSF non-convexity produces a more general result.

Definition 3. *Let $T_A(N)$ be the set of $N \times N$ matrices in $\mathbb{T}^{N \times N}$ with non-zero support A, i.e. the matrix is supported only where $A = 1$, where $\mathbb{T}$ is the complex unit circle.*

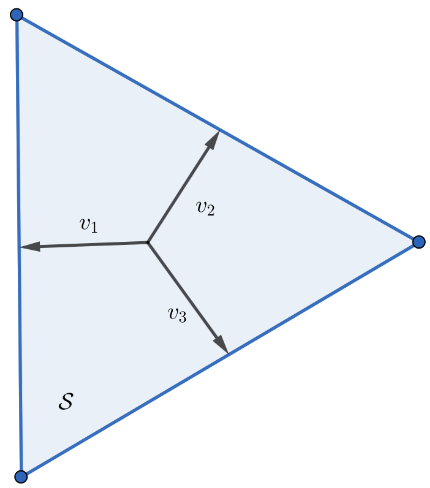


Figure 1: Example of vectors in $\mathcal{V}$. Observe that each vector v_1, v_2, v_3 is perpendicular to a side.

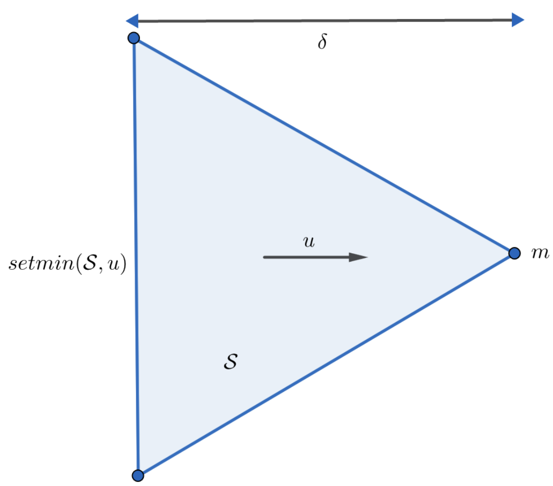


Figure 2: Example of $\mathcal{S}$ and a valid direction u. Observe that there is only one point furthest in direction u, but can be multiple points furthest in the opposite direction $-u$.

The PSF induced by a phase mask M can be modeled as the squared magnitude of the Fourier transform of the pupil function f [2].

Definition 4. *Let $f : \mathbb{R}^{N\times N} \to T_A(N)$ be defined by*

$$f(M) = A \odot \exp(iD + icM) \tag{5}$$

where $\odot$ denotes entry-wise multiplication, $D \in \mathbb{R}^{N\times N}$ and $c \in \mathbb{R} - \{0\}$ (the reals except for 0) are fixed constants, and $A \in \{0, 1\}^{N\times N}$ is the aperture.

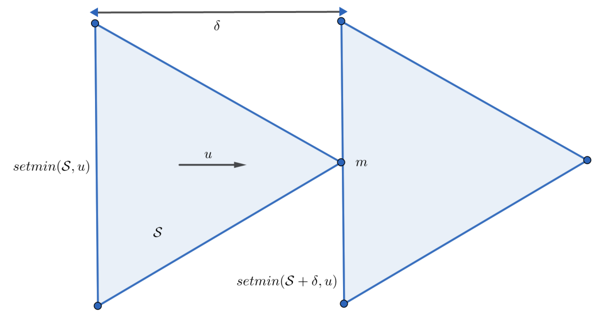


Figure 3: Example of overlap between $\mathcal{S}$ and $\mathcal{S} + \delta$.

Definition 5. *Let $g : T_A(N) \to \mathbb{R}^{N\times N}$ be defined by*

$$g(X) = \frac{|\mathcal{F}(X)| \odot |\mathcal{F}(X)|}{\|\mathcal{F}(X)\|_F^2} \tag{6}$$

where $\mathcal{F}$ denotes the discrete Fourier Transform with sufficient zero-padding, $|\cdot|$ denotes entry-wise absolute value, and $\|\cdot\|_F$ denotes the Frobenius norm.

Lemma 2. *From fourier optics theory [1], any single phase mask's PSF at a specific depth can be written as*

$$PSF = g \circ f.$$

Theorem 3. *The range of PSF is not a convex set.*

Proof. f is clearly surjective, so it suffices to argue the range of g is not convex. Assume by way of contradiction that the range of g is convex. Then, for all $X^{(1)}, \ldots, X^{(k)} \in T_A(N)$ there exists $Y \in T_A(N)$ such that $g(Y) = \frac{1}{k}\sum_{i=1}^{k} g(X^{(i)})$. By Parseval's Theorem,

$$\|\mathcal{F}(X)\|_F^2 = N^2\|X\|_F^2 = N^2 \sum_{i=0}^{N}\sum_{j=0}^{N} A_{i,j} \tag{7}$$

so the condition is

$$|\mathcal{F}(Y)| \odot |\mathcal{F}(Y)| = \frac{1}{k}\sum_{i=1}^{k} |\mathcal{F}(X^{(i)})| \odot |\mathcal{F}(X^{(i)})| \tag{8}$$

or equivalently

$$\mathcal{F}(Y) \odot \overline{\mathcal{F}(Y)} = \frac{1}{k}\sum_{i=1}^{k} \mathcal{F}(X^{(i)}) \odot \overline{\mathcal{F}(X^{(i)})}. \tag{9}$$

Then the cross-correlation theorem reduces it to

$$\mathcal{F}(Y \star Y) = \frac{1}{k}\sum_{i=1}^{k} \mathcal{F}(X^{(i)} \star X^{(i)}) \tag{10}$$

where $\star$ denotes cross-correlation. Because the Fourier Transform is linear we finally have

$$Y \star Y = \frac{1}{k} \sum_{i=1}^{k} X^{(i)} \star X^{(i)}. \tag{11}$$

Therefore, the convexity of the range of g is equivalent to the convexity of the set $\{X \star X : X \in T_A(N)\}$. We will show the set's projection onto a particular coordinate is not convex.

$$(X \star X)_{s,r} = \sum_{i=0}^{N} \sum_{j=0}^{N} X_{i,j} \overline{X_{i+s,j+r}} \tag{12}$$

where we adopt the convention that $X_{s,r} = 0$ when $s, r > N$ or $s, r < 0$. Observe that cross-correlation can be represented geometrically as shifting $\overline{X}$ over X. Let $\mathcal{S}$ be the set of coordinates with non-zero entries in X. Applying Lemma 1 to $\mathcal{S}$ shows that $\overline{X}$ and X will overlap at exactly one point. Select points $v, u \in \mathcal{S}$ such that $v - u = \delta$, then,

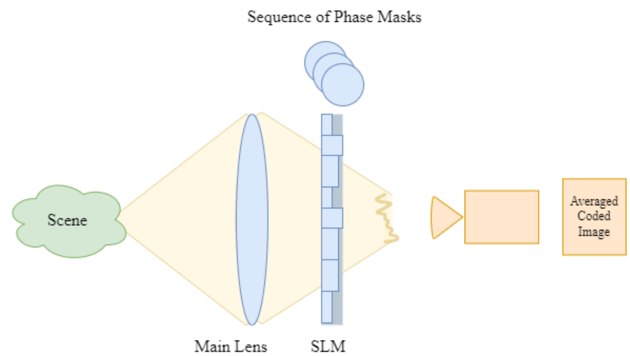
$$(X \star X)_{\delta} = X_u \overline{X_v} + \sum_{i=1}^{N^2-1} 0. \tag{13}$$

Because $X_u \overline{X_v} \in \mathbb{T}$, $(X \star X)_{\delta} \in \mathbb{T}$ which is a non-convex set. Therefore, the set of correlation's of values on the complex unit circle masked by A is also not convex. Consequently, the range of PSF is not a convex set. □

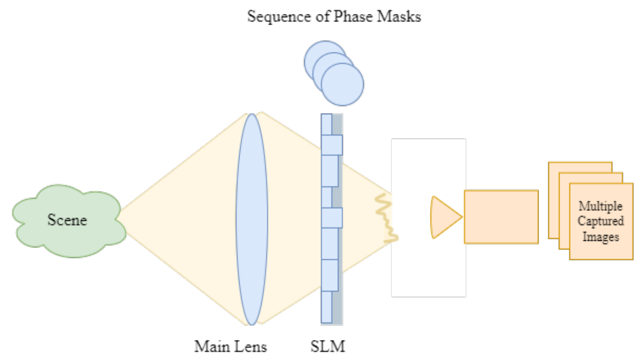
2. Discussion: Time Averaging Compared to Multi-Shot Sequences

Our optical model images a static scene through multiple phase masks which we switch between over the course of single exposure (Figure 4a). A natural question, then, is why limit ourselves to a single exposure. Why not capture a burst of images, each with a different phase mask (Figure 4b)?

While it is true that superimposing the outputs of multiple PSFs creates challenges in disambiguating outputs from phase masks, it also offers several benefits. First, because we only capture a single frame, our system uses less memory due to less I/O required. Second, imaging in a single exposure is more light efficient. Over a fixed time interval, a single exposure allows you to capture the entirety of the light from the scene. Multi-shot, alternatively, would miss photons during readout between shots.



(a) Time averaging phase masks



(b) Multi-shot phase masks

Figure 4: **Time averaging and multi-shot optical systems.** Observe that multi-shot systems capture multiple coded images, while time averaging only captures one. This means our system is more light and memory efficient.

References

[1] Joseph W. Goodman. *Introduction to fourier optics*. Freeman, 2017. 2

[2] Yicheng Wu, Vivek Boominathan, Huaijin Chen, Aswin Sankaranarayanan, and Ashok Veeraraghavan. Phasecam3d — learning phase masks for passive single view depth estimation. In *2019 IEEE International Conference on Computational Photography (ICCP)*, pages 1–12, 2019. 2