Enhancing Image Restoration Transformer via Adaptive Translation Equivariance

Supplementary Material

A. Translation equivariance v.s. translation invariance

In this section, we briefly explore the difference between translation equivariance v.s. translation invariance.

Definition A.1. Translation Equivariance. We call function $\Phi(\cdot)$ is translation equivariant to translation operation $\mathcal{T}(\cdot)$, if $\Phi(\mathcal{T}(x)) = \mathcal{T}(\Phi(x))$, where x is the input signal.

Definition A.2. Translation Invariance. We call function $\Phi(\cdot)$ is translation invariant to translation operation $\mathcal{T}(\cdot)$, if $\Phi(\mathcal{T}(x)) = \Phi(x)$, where x is the input signal.

According to the aforementioned definitions, it can be inferred that a translation-equivariant operator ensures that when the input undergoes a translation, the output is translated accordingly. Equivariance maintains the spatial correspondence between the input and output. Conversely, a translation-invariant operator guarantees that the output remains unchanged when the input is translated, thereby ensuring stability against translation. Translation invariance highlights robustness to input variations.

In the context of image restoration tasks, where precise restoration is required at the pixel level, the model should maintain a degree of translation equivariance rather than translation invariance to enhance the fidelity of the restored image as we discussed in Sec. 1.

B. Proofs

In this section, we provide proofs of Theorem 3.2 and Theorem 3.3.

Proof of Theorem 3.2.

Proof. Given a function $\Phi(x)_i$ is transformed from $x_j = [i-b, i+b]$, where b is the sliding boundary, $\Phi(x)_i$ can be rewritten as follows.

$$\Phi(x)_i = \Phi(x_{j=[i-b,i+b]}), \tag{8}$$

where j is the index of the input signal x.

Given the translation operator $\mathcal{T}(\cdot)$, which satisfies $\mathcal{T}(x_j) = x_{j+\delta}$, where δ is a constant scalar.

$$\begin{aligned}\mathcal{T}(\Phi(x)_i) &= \Phi(x)_{i+\delta} \\ &= \Phi(x_{j=[i+\delta-b,i+\delta+b]}) \\ &= \Phi(\mathcal{T}(x)_{j=[i-b,i+b]}) \\ &= \Phi(\mathcal{T}(x))_i,\end{aligned} \tag{9}$$

which completes the proof.

Proof of Theorem 3.3.

Proof. Given functions $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$ are translation equivariant to translation operation $\mathcal{T}(\cdot)$, according to Definition 3.1, we get:

$$\Phi_1(\mathcal{T}(x)) = \mathcal{T}(\Phi_1(x)), \tag{10}$$

$$\Phi_2(\mathcal{T}(x)) = \mathcal{T}(\Phi_2(x)). \tag{11}$$

Since the translation operation $\mathcal{T}(\cdot)$ is a linear operator, we can sum the above equations as follows.

$$\begin{aligned}\Phi_1(\mathcal{T}(x)) + \Phi_2(\mathcal{T}(x)) &= \mathcal{T}(\Phi_1(x)) + \mathcal{T}(\Phi_2(x)) \\ &= \mathcal{T}(\Phi_1(x) + \Phi_2(x)).\end{aligned} \tag{12}$$

It can be seen that the function $\Phi_1(\cdot) + \Phi_2(\cdot)$ is translation equivariant to translation operation $\mathcal{T}(\cdot)$.

When stacking the functions $\Phi_1(\cdot)$ and $\Phi_2(\cdot)$ in parallel,

$$\begin{aligned}\Phi_2(\Phi_1(\mathcal{T}(x))) &= \Phi_2(\mathcal{T}(\Phi_1(x)) \\ &= \mathcal{T}(\Phi_2(\Phi_1(x))).\end{aligned} \tag{13}$$

The function $\Phi_2(\Phi_1(\cdot))$ is also translation equivariant to translation operation $\mathcal{T}(\cdot)$, which completes the proof.