PLMP – Point-Line Minimal Problems for Projective SfM

Supplementary Material

A. Proofs

Here, we provide formal proofs for the lemmas and propositions used in the main paper.

Lemma A.1. *For any minimal PLP with at least two views, the underlying point arrangement is among those depicted in Tables 1 or 3.*

Proof. Since minimal PLPs are balanced, it suffices to restrict to tuples (p^f, p^d) appearing in at least one balanced PLP. For $p^f + p^d \geq 9$, there are no balanced problems. Thus, we can assume $p^f + p^d \leq 8$. Starting with $p^f + p^d = 8$, we only have to consider one tuple, namely $(2, 6)$ which is not possible by Lemma 4.1.

We consider the case $p^f + p^d = 7$. The tuples $(7, 0)$ and $(6, 1)$ give rise to the unique generic configurations represented by the first two entries of Table 1. In case of three or more collinear points, we will refer to a line containing these points as a supporting line. Lemma 4.1 implies that on each supporting line we have precisely three points. For the tuple $(5, 2)$ we thus need two distinct supporting lines. These lines can either have precisely one point in common or no point in common. In both cases we obtain a unique configuration.

Next, we have to consider $p^f = 4$ and $p^d = 3$, in which case we need 3 supporting lines by Lemma 4.1. If a point is on no line, we have three dependent points in the span of three other points which is excluded by Lemma 4.1. Thus, all points lie on one of the 3 supporting lines. First, we investigate the case of one point lying on all 3 lines. Besides this point each line must contain two further points and no other point is allowed on two of the lines as two points define a line uniquely, so the only possibility is the last entry of Table 1. Second, we consider the case where each point lies on 1 or 2 lines. As there are in total 9 points on the lines (counted with multiplicity), there are precisely 2 points, say p_1 and p_2, lying on two of the lines and the other 5 lying only on one of the lines. Exaclty one of the supporting lines contains both p_1 and p_2. As the 5 other points lie on exactly one line each and we have still 5 positions to fill, there is a unique configuration: see the penultimate entry of Table 1.

The tuples $(3, 4)$ and $(2, 5)$ are not possible by Lemma 4.1 and the tuples $(1, 6)$ and $(0, 7)$ cannot exist as we need at least two free points to define the first dependent point. Any configuration appearing for $p^f + p^d \leq 6$ is a subconfiguration of a seven-point one. Hence, the proofs are the same and we listed all such problems in Table 3. $\square$

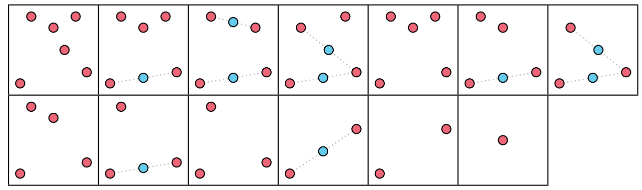


Table 3. Arrangements of at most 6 points in minimal PLPs.

Proof of Lemma 4.1. As an alternative to standard homography arguments, we illustrate the stabilizer technique carried out in Examples 4.8 and 4.10 to prove Lemma 4.1.

First, consider $\mathcal{X}$ to be the variety of two free points and $p^d \geq 1$ dependent points. By applying homographies we know that the orbit of $\mathcal{X}' := \{(e_1, e_2, e_1 + e_2)\} \times \{\lambda e_1 + \mu e_2 \mid (\lambda : \mu) \in \mathbb{P}^1\}^{p^d - 1}$ is dense in $\mathcal{X}$. The stabilizer of the first three points is given by

$$\begin{bmatrix} 1 & 0 & h_1 & h_2 \\ 0 & 1 & h_3 & h_4 \\ 0 & 0 & h_5 & h_6 \\ 0 & 0 & h_7 & h_8 \end{bmatrix}$$

and it indeed stabilizes all further dependent points. The camera matrices can generically be normalized to

$$\mathcal{C}' = \begin{bmatrix} c_1 & 1 & 1 & 0 \\ c_2 & c_3 & 1 & 0 \\ c_4 & c_5 & 1 & 0 \end{bmatrix}.$$

Now, the inequality in (7) specializes to

$$5m + p^d - 1 \geq m(5 + p^d - 1),$$

i.e., $0 \geq (m - 1)(p^d - 1)$. Since $m \geq 2$, we get $p^d \leq 1$.

For the second statement, suppose for the sake of contradiction that we have a minimal problem with $p^d > 2$ points in a plane spanned by 3 free points. We consider the subproblem consisting of the 3 free points and the first 3 dependent points. We show now that each of the 3 dependent points has to lie on exactly one of the 3 lines spanned by the 3 free points. To prove this formally, we begin with investigating how arrangements with 3 free and only 2 dependent points can look like:

By the first part of this lemma, each of the free points has to lie on at least one supporting line as otherwise both dependent points have to lie on the same supporting line. Since we only have 2 supporting lines, containing 3 points each (so in total 6 when counted with multiplicities), there is precisely one point p_1 on both supporting lines.

Adding the third dependent point, we see that its supporting line cannot contain p_1 as otherwise there would be four points on a line which we proved above to be impossible. Hence, it contains two points p_2 and p_3 such that p_1, p_2 and p_3 are not collinear. Thus, we can view p_1, p_2, p_3 as the free points and we can write the dependent points as $p_4 = \lambda_1 p_1 + \mu_1 p_2, p_5 = \lambda_2 p_1 + \mu_2 p_3$ and $p_6 = \lambda_3 p_2 + \mu_3 p_3$, where $(\lambda_i : \mu_i) \in \mathbb{P}^1$.

We can choose a homography taking p_1 to e_1, p_2 to e_2, p_3 to e_3, p_4 to $e_1 + e_2$ and p_5 to $e_1 + e_3$. Hence $\mathcal{X}' := \{(e_1, e_2, e_3, e_1 + e_2, e_1 + e_3)\} \times \{\lambda e_2 + \mu e_3 \mid (\lambda : \mu) \in \mathbb{P}^1\}$ is dense in the subproblem with precisely 3 dependent points. The stabilizer of $\mathcal{X}'$is given by

$$\mathrm{Stab}(\mathcal{X}') = \begin{bmatrix} 1 & 0 & 0 & h_1 \\ 0 & 1 & 0 & h_2 \\ 0 & 0 & 1 & h_3 \\ 0 & 0 & 0 & h_4 \end{bmatrix}.$$

The camera matrices can then generically be normalized to

$$\mathcal{C}' = \begin{bmatrix} c_1 & c_2 & 1 & 0 \\ c_3 & c_4 & c_5 & 0 \\ c_6 & c_7 & c_8 & 0 \end{bmatrix}.$$

Now, the inequality in (7) specializes to

$$8m + 1 \geq m(3 \cdot 2 + 3),$$

i.e., $1 \geq m$, contradicting our assumption that $m \geq 2$. □

Proof of Proposition 4.9. First we reduce to the case $p' = p$ and $l' = l$. Consider the diagram

$$\begin{tikzcd} (\mathcal{C}^m \times \mathcal{X}_{p,l,\mathcal{I}})/\mathrm{PGL}_4 \arrow[r, dashed, "\Phi_{p,l,\mathcal{I},m}"] \arrow[d] & \mathcal{Y}^m_{p,l,\mathcal{I}} \arrow[d, two heads] \\ (\mathcal{C}^m \times \mathcal{X}_{p',l',\mathcal{I}'})/\mathrm{PGL}_4 \arrow[r, dashed, "\Phi_{p',l',\mathcal{I}',m}"] & \mathcal{Y}^m_{p',l',\mathcal{I}'} \end{tikzcd}$$

where the unlabled vertical maps are given by projecting onto the points and lines present in the subarrangement. The diagram commutes, and the upper and right morphisms have full dimensional images, so has their composition. Thus, the same holds for $\Phi_{p',l',\mathcal{I}',m}$. Hence, we can assume without loss of generality that $\mathcal{X}_{p',l',\mathcal{I}'} = \mathcal{X}_{p,l,\mathcal{I}}$.

Now we consider $\Phi : \mathcal{C}^m \times \mathcal{X}' \to \mathcal{Y}^m_{p,l,\mathcal{I}}$ given by $(P_1, ..., P_m, A) \mapsto (P_1 A, ..., P_m A)$. We claim that this map has also full dimensional image. This follows from the diagram

$$\begin{tikzcd} \mathcal{C}^m \times \mathcal{X}' \arrow[rr, dashed, "\iota"] \arrow[dr, dashed, "\Phi"'] & & (\mathcal{C}^m \times \mathcal{X}_{p,l,\mathcal{I}})/\mathrm{PGL}_4 \arrow[dl, dashed, "\Phi_{p,l,\mathcal{I},m}"] \\ & \mathcal{Y}^m_{p,l,\mathcal{I}} & \end{tikzcd}$$

where ι takes a tuple of cameras and points and lines to their equivalence class on the right. Since $\mathrm{PGL}_4 \cdot \mathcal{X}'$ has the same dimension as $\mathcal{X}_{p,l,\mathcal{I}}$, the composition has full dimensional image and thus Φ has by commutativity.

Finally, since $(\mathcal{C}' \cdot \mathrm{Stab}(\mathcal{X}'))^m$ has the same dimension as $\mathcal{C}^m$, we can restrict Φ to this subset without reducing the dimension of its image. By definition of the stabilizer, the image of Φ on this subset agrees with the image of $\Phi^{\mathrm{red}}_{p',l',\mathcal{I}',m}$ finishing the proof. The inequality (7) follows directly from the Fiber-Dimension Theorem. □

B. Non-minimality via subproblem stabilizers

In Section 4.3, we explained a non-minimality criterion to formally disprove the minimality of a balanced PLP, focusing at one concrete example problem. Here, we explain how that strategy shows the non-minimality for in total 130 balanced PLPs (out of the $149 = 434 - 285$ balanced problems that are claimed to be non-minimal in part c) of the proof of our Main Theorem 2.3). For that, we systematically consider sub-arrangements that appear in balanced PLPs, to identify relevant reduced sub-problems $(\mathcal{C}', \mathcal{X}')$ as in Definition 4.7. All reduced subproblems that we need to consider in this section are listed in Table 4. There, we use ℓ^a_* to denote the number of lines adjacent to a single (fixed) point. The diagram in the first column shows at which point these lines are attached (which is relevant for the second to last row where not all points are indistinguishable due to some being collinear and some not). The diagram itself does not depict all adjacent lines but only those that are normalized; this is indicated by the inequality in the description of $\mathcal{X}'$. For example, we can consider Example 4.8 and Example 4.12 both as instances of the first row of Table 4.

Similarly to Example 4.10 and the proof of Lemma 4.1 in SM Section A, we can apply inequality (7) to the reduced subproblems in Table 4 to extract neccessary conditions that minimal problems have to satisfy. These conditions are listed in Table 5. The table shall be read as follows: Given a minimal PLP with one of the depicted arrangements as a subproblem, then the last three columns specify how many lines can at most be attached to the point specified by the diagram depending on the number of cameras. For example, the first row of Table 5 says that for three cameras, at most 7 lines can be attached to the point, while for four cameras, there are at most 6 attached lines, and in case of five or more cameras, at most 5 lines can be attached to a single point.

Proposition B.1. *Any minimal PLP has to satisfy all criteria listed in Table 5. In other words, if a given PLP has a subproblem as shown in the first entry of a row, it has to satisfy the given bound on the number of lines attached to a single point, as specified in the last three columns depending on the number of cameras.*

Proof. The criteria 1 to 4 follow immediately from the first

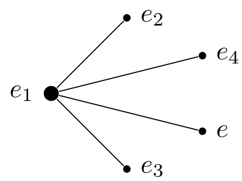
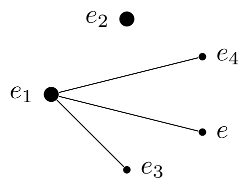
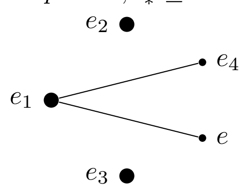
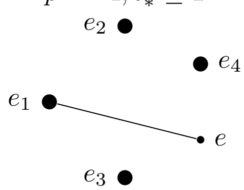
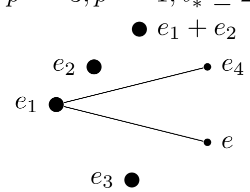
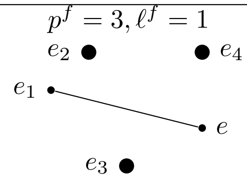
$\mathcal{X}'$	$\mathrm{Stab}(\mathcal{X}')$	$\mathcal{C}'$
$p^f = 1, \ell_*^a \geq 4$ 	$\begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 \\ c_1 & c_2 & c_3 & c_4 \\ c_5 & c_6 & c_7 & 1 \end{bmatrix}$
$p^f = 2, \ell_*^a \geq 3$ 	$\begin{bmatrix} \lambda_1 & 0 & \lambda_2 & \lambda_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 \\ c_5 & c_6 & c_7 & c_8 \end{bmatrix}$
$p^f = 3, \ell_*^a \geq 2$ 	$\begin{bmatrix} \lambda_1 & 0 & 0 & \lambda_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & c_1 & 0 \\ c_2 & c_3 & c_4 & c_5 \\ c_6 & c_7 & c_8 & c_9 \end{bmatrix}$
$p^f = 4, \ell_*^a \geq 1$ 	$\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & c_1 & c_2 \\ c_3 & c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 & c_{10} \end{bmatrix}$
$p^f = 3, p^d = 1, \ell_*^a \geq 2$ 	$\begin{bmatrix} 1 & 0 & 0 & \lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} c_1 & c_2 & 1 & 0 \\ c_3 & c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 & c_{10} \end{bmatrix}$
$p^f = 3, \ell^f = 1$ 	$\begin{bmatrix} \lambda & 0 & 0 & 0 \\ \mu & 1 & 0 & 0 \\ \mu & 0 & 1 & 0 \\ \mu & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & c_1 & c_2 & c_3 \\ 1 & c_4 & c_5 & c_6 \\ 1 & c_8 & c_8 & c_9 \end{bmatrix}$

Table 4. Reduced sub-PLPs. e_i denote standard basis vectors of $\mathbb{R}^4$, $e := e_2 + e_3 + e_4$, and ℓ_*^a denotes the number of lines adjacent to e_1.

four rows of Table 4. For criterion 5, we observe that any matrix in the stabilizer of the subproblem without the dependent point already fixes the dependent point and thus the stabilizer does not change when adding it. Thus, in the third row of Table 4, we can add an unnormalized dependent point on the line spanned by e_2 and e_3 to turn $\mathcal{X}'$ into another reduced subproblem $(\mathcal{X}'', \mathcal{C}')$, with the same camera variety $\mathcal{C}'$ as in row 3 of Table 4. Thus, we have $\dim(\mathcal{C}') = 9$ and $\dim(\mathcal{X}'') = 1 + \dim(\mathcal{X}') = 1 + 2 \cdot (\ell_*^a - 2)$, where the summand of 1 comes from the unnormalized dependent point. Lastly, $\dim(\mathcal{Y}_{p',l',\mathcal{I}'}) = 2 \cdot 3 + 1 + \ell_*^a$. Hence, in this case, inequality (7) becomes

$$2 \cdot (m-2) + 1 \geq (m-2) \cdot l_*^a, \tag{9}$$

which yields the claimed bounds for $m = 3$ and $m > 3$. Criterion 6 is obtained similarly using the fourth row of Table 4. For criterion 7, we use the same argument together with the fifth row of Table 4. $\square$

To deal with free lines in the case of five points in one plane, we can formulate an 8th criterion which is independent of the number of cameras as long as this number is greater than 2.

Lemma B.2. *Let $(p, \ell, \mathcal{I}, m)$ be a minimal PLP with $m \geq 3$ and 5 points contained in a plane, 2 out of which are dependent on the other 3. Then, any line has to be adjacent to one of these 5 points.*

Proof. Suppose there is a line not adjacent to one of the 5 points. We consider the reduced subproblem containing the 3 free points in the plane and the line. Note that the line is free in the subproblem, even if it was not free in the original PLP. A description of the stabilizers and reduced varieties is shown in the last row of Table 4. We observe that the stabilizer acts as the identity on the plane spanned by the free points and thus the two dependent points are also fixed. Hence, as in the proof of Proposition B.1, by adding the unnormalized dependent points to $\mathcal{X}'$, we obtain $\mathcal{X}''$ with the same stabilizer. Next, we compute $\dim(\mathcal{C}') = 9$, $\dim(\mathcal{X}'') = 2$ and $\dim(\mathcal{Y}_{p',l',\mathcal{I}'}) = 2 \cdot 3 + 2 + 2$. Therefore, inequality (7) becomes $9m + 2 \geq 10m$, i.e., $m \leq 2$. $\square$

Remark B.3. There is also a geometric proof of this lemma. After normalizing the world points, each camera gives a unique isomorphism between the plane Π spanned by the points and the image plane. The intersection of the preimage plane of the line under the camera map with the plane Π is given by applying the inverse of the isomorphism to the line in the image. Hence, for each image, we get a unique line in the plane Π. These three lines have to intersect in one point (namely the intersection of the true world line with the plane Π), but three generic lines do not intersect and hence the reconstruction is not possible in general. $\diamondsuit$

The 130 non-minimal balanced PLPs mentioned above are listed in our code[1](lines 1396-1544), together with the number of one neccessary minimality-condition they violate.

C. Non-minimality via elimination

In the previous section, we described a method to identify overconstrained subproblems in non-minimal problems. This method covered the majority of the non-minimal balanced problems described in part c) of the proof of our main theorem. However, there are 19 cases left for which we need a more detailed look at the equations. But also here we will make use of the stabilizers. In order to identify contradictory constraints, we start of by finding a minimal subproblem to eliminate some of the variables. This factorizes the original non-minimal problem into the minimal subproblem and the 'remainder problem', analogously to the factorization in Example 4.12. We then find contradictory equations in the 'remainder problem'.

Example C.1. One of the remaining balanced PLPs is the scenario where 3 cameras observe 4 free points, 1 dependent point, 1 free line and 2 lines attached to a point that is not among the collinear points; see first entry of Table 6. We consider the subproblem arising from omitting the free line. More explicitely, we choose the normalized point-line variety to be $\mathcal{X}' = \{(e_1, e_2, e_3, e_4, \overline{e_1 e})\} \times \{\lambda e_3 + \mu e_4 \mid (\lambda : \mu) \in \mathbb{P}^1\} \times \{\overline{e_1 Q} \mid Q = \lambda e_2 + \mu e_3 + \nu e_4, (\lambda : \mu : \nu) \in \mathbb{P}^2\}$. We can compute its stabilizer to be

$$\left\{ \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| \lambda \neq 0 \right\}. \tag{10}$$

Setting the reduced camera variety to

$$\mathcal{C}' := \left\{ \begin{bmatrix} 1 & 1 & c_1 & c_2 \\ c_3 & c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 & c_{10} \end{bmatrix} \middle| c_i \in \mathbb{R} \right\}, \tag{11}$$

we can verify that this indeed satisfies the definition of a reduced subproblem. Moreover, it is a balanced subproblem, i.e., it satisfies the inequality in (7) with equality. To see this, we just compute the dimensions: $\dim(\mathcal{C}') = 10$, $\dim(\mathcal{X}') = 1 + 2$ and $\dim(\mathcal{Y}_{p',l',\mathcal{I}'}) = 4 \cdot 2 + 1 + 2 \cdot 1$, and indeed $30 + 3 = 3 \cdot 11$. Hence, if we assume for the sake of contradiction that the original PLP was minimal, then Proposition 4.9 (together with Lemma 4.3) would tell us that the subproblem is minimal as well.

Therefore, we could solve the subproblem first and then assume that the camera parameters c_j in (11) are just numbers. To distinguish between the different cameras, we decorate them with a superscript i. Re-introducing the parameters that we could not reconstruct from the subproblem, we obtain cameras of the form

$$\begin{bmatrix} \lambda^i & 1 & c_1^i & c_2^i \\ c_3^i \lambda^i & c_4^i & c_5^i & c_6^i \\ c_7^i \lambda^i & c_8^i & c_9^i & c_{10}^i \end{bmatrix}. \tag{12}$$

Here, the remaining PGL_4-action allows us to set $\lambda^1 = 1$.

Now, we consider the free line from the original PLP. Generically, it intersects the plane orthogonal to e_1 at a unique point of the form $\begin{bmatrix} 0 & 1 & x_1 & x_2 \end{bmatrix}^\top$. After applying C_i to it, the resulting image point has to be orthogonal to the coefficient vector representing the free line in the i-th image: $\begin{bmatrix} y_1^i & y_2^i & 1 \end{bmatrix}^\top$. Since our point on the line has a 0 in its first entry, the resulting equation

$$\begin{bmatrix} y_1^i & y_2^i & 1 \end{bmatrix} C_i \begin{bmatrix} 0 & 1 & x_1 & x_2 \end{bmatrix}^\top = 0$$

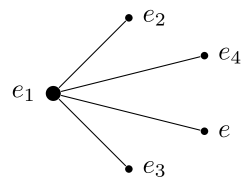
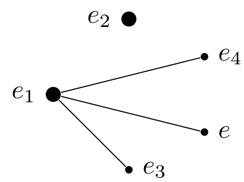
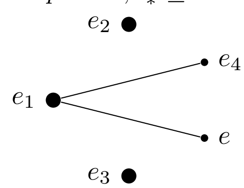
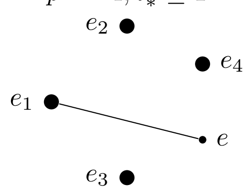
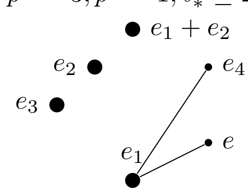
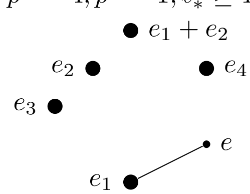
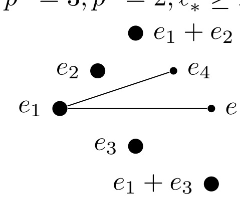
Arrangement	Number	$m=3$	$m=4$	$m \in \{5,6,7,8\}$
$p^f=1, \ell^a_* \geq 4$ 	1	$\ell^a_* \leq 7$	$\ell^a_* \leq 6$	$\ell^a_* \leq 5$
$p^f=2, \ell^a_* \geq 3$ 	2	$\ell^a_* \leq 6$	$\ell^a_* \leq 5$	$\ell^a_* \leq 4$
$p^f=3, \ell^a_* \geq 2$ 	3	$\ell^a_* \leq 5$	$\ell^a_* \leq 4$	$\ell^a_* \leq 3$
$p^f=4, \ell^a_* \geq 1$ 	4	$\ell^a_* \leq 4$	$\ell^a_* \leq 3$	$\ell^a_* \leq 2$
$p^f=3, p^d=1, \ell^a_* \geq 2$ 	5	$\ell^a_* \leq 3$	$\ell^a_* \leq 2$	$\ell^a_* \leq 2$
$p^f=4, p^d=1, \ell^a_* \geq 1$ 	6	$\ell^a_* \leq 2$	$\ell^a_* \leq 1$	$\ell^a_* \leq 1$
$p^f=3, p^d=2, \ell^a_* \geq 2$ 	7	$\ell^a_* \leq 3$	$\ell^a_* \leq 2$	$\ell^a_* \leq 2$

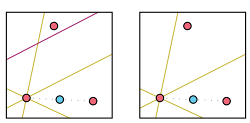
Table 5. Necessary conditions for minimality. Same notation as in Table 4.

does not contain λ^i. Hence, for $i = 1, 2, 3$, we obtain three independent equations in two variables, which has no solution generically. $\diamondsuit$

The strategy used in this example of eliminating variables and finding an overconstrained subsystem can directly be used for 15 further cases. These are listed in Table 6 together with the respective subarrangements one has to consider and the parts of the 3D arrangement causing the contradictory constraints.

Among the three remaining non-minimal balanced PLPs, there is one further case which can be solved using this strategy only requiring a minor adaptation of the argument. Instead of considering a minimal subproblem, we consider a subproblem with a two-dimensional solution set. This, however, does not affect the possibility of splitting the system of equations into one overconstrained subproblem and an underconstrained subproblem.

Example C.2. We consider the balanced PLP where 5 cameras observe 3 free points, 1 dependent point, 1 free line and 3 lines attached to a collinear point. We study the sub-PLP arising from omitting the free line, as shown on the right:



We choose the following normalized point-line variety $\mathcal{X}' := \{(e_1, e_2, e_3, e_1{+}e_2, \overline{e_1 e_4}, \overline{e_1 e})\} \times \{\overline{e_1 Q} \mid Q = \lambda e_2 + \mu e_3 + \nu e_4, (\lambda : \mu : \nu) \in \mathbb{P}^2\}$, with stabilizer and reduced camera variety

$$\mathrm{Stab}(\mathcal{X}') = \left\{ \begin{bmatrix} 1 & 0 & 0 & \lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}, \; \mathcal{C}' := \left\{ \begin{bmatrix} c_1 & c_2 & 1 & 0 \\ c_3 & c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 & c_{10} \end{bmatrix} \right\}. \quad (13)$$

We assume for contradiction that the original PLP is minimal. Then, by Proposition 4.9, the reduced joint camera map has a full-dimensional image. Thus, the Fiber-Dimension theorem tells us that its generic fiber has dimension $m \cdot \dim \mathcal{C}' + \dim \mathcal{X}' - m \cdot \dim \mathcal{Y} = 5 \cdot 10 + 2 - 5 \cdot 10 = 2$. To consider the free line from the original PLP, we reintroduce the stabilizer, which results in cameras of the form

$$\begin{bmatrix} c_1^i & c_2^i & 1 & c_1^i \lambda^i \\ c_3^i & c_4^i & c_5^i & c_3^i \lambda^i + c_6^i \\ c_7^i & c_8^i & c_9^i & c_7^i \lambda^i + c_{10}^i \end{bmatrix}, \quad (14)$$

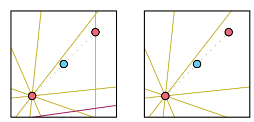
where $\lambda^1 = 1$ to fix the PGL_4-action. The only difference to Example C.1 is that we do not have a finite set of these c_j^i but that they form a 2-dimensional set. Now we consider the points $\begin{bmatrix} x_1^1 & x_2^1 & 1 & 0 \end{bmatrix}^\top$ and $\begin{bmatrix} x_1^2 & x_2^2 & 0 & 1 \end{bmatrix}^\top$ on the generic free line with coefficient vector $\begin{bmatrix} y_1^i & y_2^i & 1 \end{bmatrix}^\top$ on the i-th image. They give rise to the remaining 10 equations:

$$\begin{bmatrix} y_1^i & y_2^i & 1 \end{bmatrix} C_i \begin{bmatrix} x_1^j & x_2^j & 0 & 1 \end{bmatrix}^\top = 0$$

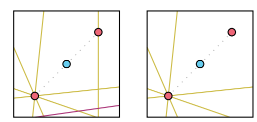
for i from 1 to 5 and $j = 1, 2$. The variables λ^i for i from 2 to 5 and x_1^2, x_2^2 do not appear within the first five constraints, and thus these variables are underconstrained. Hence, if the original PLP has a solution, it must have infinitely many, which contradicts the minimality assumption. $\diamondsuit$

For the last two balanced problems, we need to take a more detailed look in order to understand the structure of their equation systems and identify the overconstrained and underconstrained subsystems. Fortunately, the equations in both cases are very similar and we can take care of them simultaneously.

Example C.3. The first scenario consists of 4 cameras observing 2 free points, 1 dependent point, 1 free line, and 6 adjacent lines, 5 of which are attached to a single point. In the other case, we have 6 cameras observing 2 free points, 1 dependent point, 1 free line and 5 adjacent lines, 4 of which are attached to a single point. In either case, we consider the subproblem that arises from omitting the free line and the single attached line.



(2,1,1,6)



(2,1,1,5)

We will focus on the smaller example with 4 cameras, but all computations work the same for the 6-camera case. We will point it out whenever a difference occurs.

We choose the normalized point-line variety to be $\mathcal{X}' := \{e_1, e_2, e_1 + e_2, \overline{e_1 e_3}, \overline{e_1 e_4}, \overline{e_1 e}\} \times \{(\overline{e_1 Q_1}, \overline{e_1 Q_2}) \mid Q_i = \lambda_i e_2 + \mu_i e_3 + \nu_i e_4, (\lambda_i : \mu_i : \nu_i) \in \mathbb{P}^2\}$ (in the 6-camera case, omit Q_2), with stabilizer and reduced camera variety

$$\mathrm{Stab}(\mathcal{X}') = \left\{ \begin{bmatrix} 1 & 0 & \lambda & \mu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}, \; \mathcal{C}' := \left\{ \begin{bmatrix} c_1 & 1 & 0 & 0 \\ c_2 & c_3 & c_4 & c_5 \\ c_6 & c_7 & c_8 & c_9 \end{bmatrix} \right\}. \quad (15)$$

We assume again for contradiction that the original PLP is minimal. Then, for 4 cameras the reduced subproblem is balanced and minimal, whereas in case of 6 cameras the fiber of the reduced joint camera map is 2-dimensional. Reintroducing the parameters from the stabilizer, we obtain cameras of the form

$$\begin{bmatrix} c_1^i & 1 & c_1^i \lambda^i & c_1^i \mu^i \\ c_2^i & c_3^i & c_2^i \lambda^i + c_4^i & c_2^i \mu^i + c_5^i \\ c_6^i & c_7^i & c_6^i \lambda^i + c_8^i & c_6^i \mu^i + c_9^i \end{bmatrix} \quad (16)$$

and we can assume $\lambda^1 = \mu^1 = 0$ to fix the PGL_4-action. Let the generic free line be spanned by the two points $\begin{bmatrix} x_1 & x_2 & 1 & 0 \end{bmatrix}^\top$ and $\begin{bmatrix} x_3 & x_4 & 0 & 1 \end{bmatrix}^\top$. The last adjacent line was attached to the second point which is now e_2. Thus, it can be represented by one further point of the form $\begin{bmatrix} x_5 & 0 & 1 & x_6 \end{bmatrix}^\top$. As there is no immediate over-

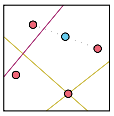
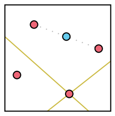
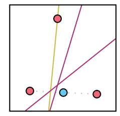
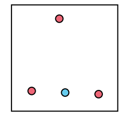
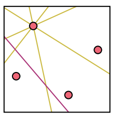
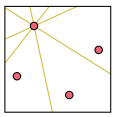
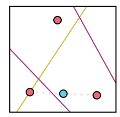
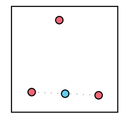
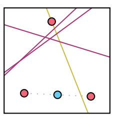
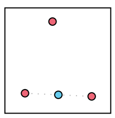
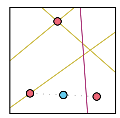
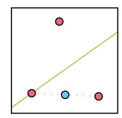
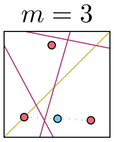
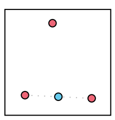
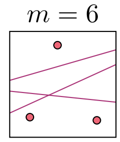
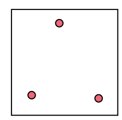
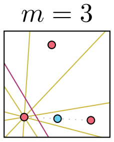
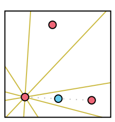
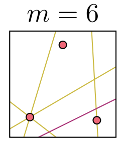
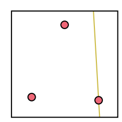
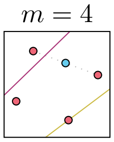
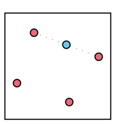
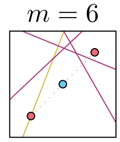
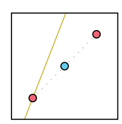
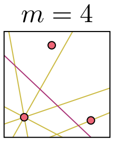
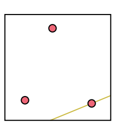
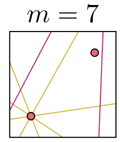
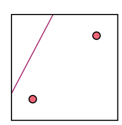
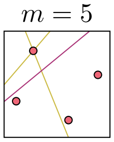
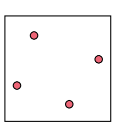
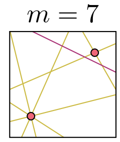
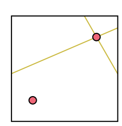
Problem	Subproblem	Constraints	Problem	Subproblem	Constraints
$m = 3$  (4,1,1,2)	 (4,1,0,2)	1 point on free line with 0 in one coordinate	$m = 5$  (3,1,2,1)	 (3,1,0,0)	1 point on each free line with 0 in one coordinate
$m = 3$  (4,0,1,4)	 (4,0,0,4)	1 point on free line with 0 in one coordinate	$m = 5$  (3,1,2,1)	 (3,1,0,0)	1 point on each free line with 0 in one coordinate
$m = 3$  (3,1,3,1)	 (3,1,0,0)	1 point on each free line with 0 in one coordinate	$m = 5$  (3,1,1,3)	 (3,1,0,1)	1 point on each unused (free and adjacent) line with 0 in one coordinate
$m = 3$  (3,1,3,1)	 (3,1,0,0)	1 point on each free line with 0 in one coordinate	$m = 6$  (3,0,3,0)	 (3,0,0,0)	1 point on each free line with 0 in one coordinate
$m = 3$  (3,1,1,5)	 (3,1,0,5)	1 point on free line with 0 in one coordinate	$m = 6$  (3,0,1,4)	 (3,0,0,1)	1 point on each unused (free and adjacent) line with 0 in one coordinate
$m = 4$  (4,1,1,1)	 (4,1,0,0)	1 point on each unused (free and adjacent) line with 0 in one coordinate	$m = 6$  (2,1,3,1)	 (2,1,0,1)	1 point on each free line with 0 in one coordinate
$m = 4$  (3,0,1,5)	 (3,0,0,1)	1 point on each unused (free and adjacent) line with 0 in one coordinate	$m = 7$  (2,0,2,4)	 (2,0,1,0)	1 point on each unused (free and adjacent) line with 0 in one coordinate
$m = 5$  (4,0,1,2)	 (4,0,0,0)	1 point on each unused (free and adjacent) line with 0 in one coordinate	$m = 7$  (2,0,1,6)	 (2,0,0,2)	1 point on each unused (free and adjacent) line with 0 in one coordinate

Table 6. List of non-minimal subproblems with overconstrained subsystems after elimination of the given subproblem. More details can be found in Example C.1 which is the first entry of this table.

or underconstrained subsystem, we study the equations directly. Let the lines on the i-th image be represented by the coefficient vectors $\begin{bmatrix} y_1^i & y_2^i & 1 \end{bmatrix}^\top$ for the free line and $\begin{bmatrix} 1 & z_1^i & z_2^i \end{bmatrix}^\top$ for the adjacent line. Define

- $\alpha_i := c_1^i y_1^i + c_2^i y_2^i + c_6^i$
- $\beta_i := y_1^i + c_3^i y_2^i + c_7^i$

- $\gamma_i := c_4^i y_2^i + c_8^i$
- $\delta_i := c_5^i y_2^i + c_9^i$
- $\varphi_i := c_1^i + c_2^i z_1^i + c_6^i z_2^i$
- $\psi_i := z_1^i c_5^i + z_2^i c_9^i$
- $\varepsilon_i := z_1^i c_4^i + z_2^i c_8^i$

Now we can write the constraints coming from the lines as

$$0 = \alpha_i x_1 + \beta_i x_2 + \alpha_i \lambda^i + \gamma_i \tag{17}$$

$$0 = \alpha_i x_3 + \beta_i x_4 + \alpha_i \mu^i + \delta_i \tag{18}$$

$$0 = \varphi_i x_5 + \psi_i x_6 + \varphi_i \lambda^i + \varphi_i \mu^i x_6 + \varepsilon_i \tag{19}$$

where i reaches from 1 to 4 resp. 6. The main idea is now to use the linearity of these constraints together with the plethora of common coefficients to find the desired subsystem. Using $\lambda^1 = \mu^1 = 0$, we can eliminate the variables x_2, x_4 and x_5. Next, Equations (17) and (18) for $i > 1$ allow to write λ^i and μ^i linearly in terms of x_1 and x_3, respectively. Inserting this into the $i-1$ last Equations (19), we obtain equations of the form $0 = a_i' x_6 + c_i' x_1 + c_i' x_3 x_6 + b_i'$. Rescaling each of these equations by the inverse of c_i' yields 3 resp. 5 equations of the form

$$0 = a_i x_6 + x_1 + x_3 x_6 + b_i. \tag{20}$$

Taking differences of these yields 2 resp. 4 equations constraining x_6 (and the 2-dimensional fiber). But in either case, this corresponds to an overconstrained subsystem, and assuming a solution to this yields a one-dimensional set of possible solutions for x_1 and x_3 in Equation (20). As in Example C.2, this contradicts the minimality of the original PLP. $\diamondsuit$

Together with the criteria derived in Section B, the 19 cases discussed in this section formally prove the completeness of the list of minimal problems given in Theorem 2.3.

D. Efficient factorizations

In Remark 4.11, we explained how to use the stabilizer technique to identify a subproblem and factorize a given minimal problem into two parts, both of which can be easier to solve than the original problem. In Example 4.12, we have seen that the degree of the problem can factorize by this method. This degree factorization is a good indicator for making a problem easier, but it is not the only way. In the following very similar example (in fact, it is the 4-camera analog to Example 4.12), we factorize a minimal problem without factorizing its degree and obtain a drastic computational speed-up since the obtained subproblems have much less unknowns.

Example D.1. We consider the scenario where 4 cameras observe 6 lines attached to a single point and additionally three free lines. This is the first example with four cameras reported in Section F. Computing the degree of this problem using Gröbner bases turned out to be particularly challenging, although it is has only degree 2. Both the naive implementation of the equations and the implementation eliminating the variables of world lines (which was the method of choice in [12]) have not terminated within a reasonable amount of time. However, it turned out that factorizing the computation into solving at first the subproblem arising from omitting the free lines and then adding them afterwards returns the degree immediately. We use basically the same normalization as in Example 4.8 and Example 4.12: $\mathcal{X}' := \{(e_1, \overline{e_1 e_2}, \overline{e_1 e_3}, \overline{e_1 e_4}, \overline{e_1 e}, \overline{e_1 Q_1}, \overline{e_1 Q_2}) \mid Q_i \neq e_1\}$ and also the stabilizers and reduced cameras stay the same. This reduced subproblem is minimal of degree 2. After solving it, one has to add the remaining 3 free lines again and thus needs to reintroduce the stabilizer exactly as in Equation (5). The PGL_4-action can be fixed afterward by setting $\lambda_j^1 = 1$. This remainder problem in the free lines and the missing camera parameters has degree one. An explicit implementation of this example can be found in the attached code[1] (lines 1805-1953). $\diamondsuit$

Analogously, one can find for many more minimal problems minimal subproblems that allow for a tremendous speedup in computation time.

E. Camera registration

The two PLPs from Theorem 2.3 (b) that are minimal for arbitrarily many cameras are in fact also minimal for a single view. Moreover, there are a few other minimal problems for a single projective camera:

Proposition E.1. *All minimal PLPs for a single projective camera are depicted in Table 7. The five right-most PLPs are infinite families as they admit an arbitrary number of (non-depicted) dependent points. All PLPs have degree* 1.

To prove this, we first classify the balanced problems. For a single camera, the corresponding dimension equation becomes

$$\begin{aligned} 11 + 3p^f + p^d + 4\ell^f + 2\ell^a - 15 &= 2p^f + p^d + 2\ell^f + \ell^a \\ \iff p^f + 2\ell^f + \ell^a &= 4. \end{aligned} \tag{21}$$

Each of the appearing parameters above is upper bounded by 4, and so we can perform a simple case distinction to obtain the 8 possible cases without dependent points shown in Table 7.

Since dependent points do not appear in the equation, there is an infinite family of possible problems for each of these PLPs with at least two free points. However, we still have to see that these are indeed all problems, i.e., that we can assume without loss of generality that adjacent lines are attached to free points.

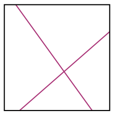
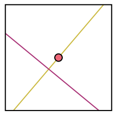
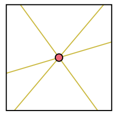
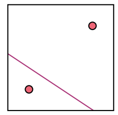
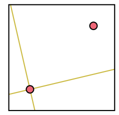
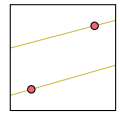
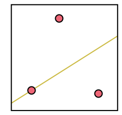
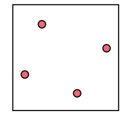
m	(p^f, p^d, l^f, l^a), algebraic degree							
1	 (0,0,2,0), **1**	 (1,0,1,1), **1**	 (1,0,0,3), **1**	 (2,0,1,0), **1**	 (2,0,0,2), **1**	 (2,0,0,2), **1**	 (3,0,0,1), **1**	 (4,0,0,0), **1**

Table 7. Single-view minimal PLPs with their associated degree.

Lemma E.2. *1. In the case of two free points, any two distinct points allow to define the other points of the arrangement as dependent points.*

2. In the case of three free points, it holds that for any point of the arrangement we can find two further points such that these three points allow to define any other point of the arrangement as a dependent point.

Proof. The first statement follows directly from the fact that all points lie on the line spanned by the two free points and any two distinct points on a line define it uniquely.

For the other statement let X be any point. Assume first that X lies on a line spanned by two of the three free points. In that case, we can take any other point on this line and the third free point since we can then construct all three of the original free points as dependent points and thus the statement follows. Otherwise, we know that the line used to define X does not contain two of the free points but the arrangement contains at least one such line since the definition of the first dependent point requires a line through two free points. Since all these lines sit inside the span of the three free points (i.e., a plane), they all intersect. Let Y be the intersection point of the supporting line used to define X and a line L containing two free points. Since we assume that all intersections of lines are present in the arrangement, so is Y. Now we pick as a third point one of the two originally free points on L. Since X does not lie on a line spanned by two free points, it is impossible that all dependent points lie on one line, and thus there is a supporting line containing the last free point. This line intersects now both the line used to define X and L, and thus we have at least two points on it and can thus construct the last originally free point as a dependent point on that line. □

Using the previous lemma we see that we can assume without loss of generality that all lines are attached to free points since we can simply re-choose the free points in the arrangement to be the points where lines are attached. Using this, the following result allows to reduce to the case of no dependent point.

Lemma E.3. *Let $\mathcal{P}$ be a minimal PLP of degree d for a single camera. Then also the PLP $\mathcal{P}'$ arising from $\mathcal{P}$ by adding a single dependent point is minimal of degree d.*

Proof. Consider the d solutions $(P_1, A_1), \ldots, (P_d, A_d)$ of the problem $\mathcal{P}$. We show that each of these admits a unique extension to $\mathcal{P}'$. This proves the desired result since each solution to $\mathcal{P}'$ projects onto a solution of $\mathcal{P}$.

On the given image, the dependent point lies on a line between two points x_1 and x_2. Those correspond to world points X_1 and X_2 in the i-th solution (P_i, A_i). For generic input images, the center of the camera P_i does not lie on the line $\overline{X_1 X_2}$. Thus, the camera P_i yields an isomorphism between the lines $\overline{X_1 X_2}$ and $\overline{x_1 x_2}$. Hence, the dependent point on $\overline{x_1 x_2}$ corresponds to a unique world point on $\overline{X_1 X_2}$. □

Putting everything together, we can now show that Table 7 indeed shows all single-view minimal PLPs.

Proof of Proposition E.1. Considering (21) together with Lemma E.2 yields that Table 7 lists all balanced PLPs for a single projective view, up to arbitrarily many dependent points for the five right-most PLPs. Due to Lemma E.3, it is sufficient to prove minimality (and compute the degree) for the balanced PLPs without any dependent points. This can be done analogously to the proof of Theorem 2.3 (b) in Section 4.2. A list with the respective normalizations can be found in Table 8. From the chosen normalizations it follows directly that the reconstruction problem becomes a linear problem since all appearing equations are linear. □

Remark E.4. The two minimal PLPs from Theorem 2.3 (b) that work for arbitrarily many views are special cases of the right-most PLP in Table 7. Note that, differently from the normalizations in the proof above, the normalizations in the proof of Theorem 2.3 (b) did not affect the camera parameters (only the point arrangements). Therefore, the two PLPs from Theorem 2.3 (b) are minimal for an arbitrary number of cameras as each camera can be registered independently using the same world coordinate system. ♢

F. Minimal PLPs

The 285 minimal PLPs from Theorem 2.3 (c) are depicted in Tables 9 – 12.

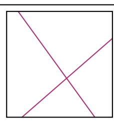
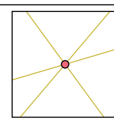
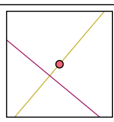
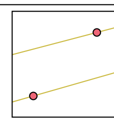
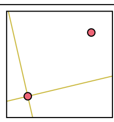
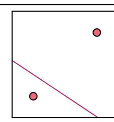
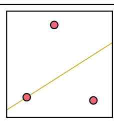
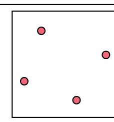
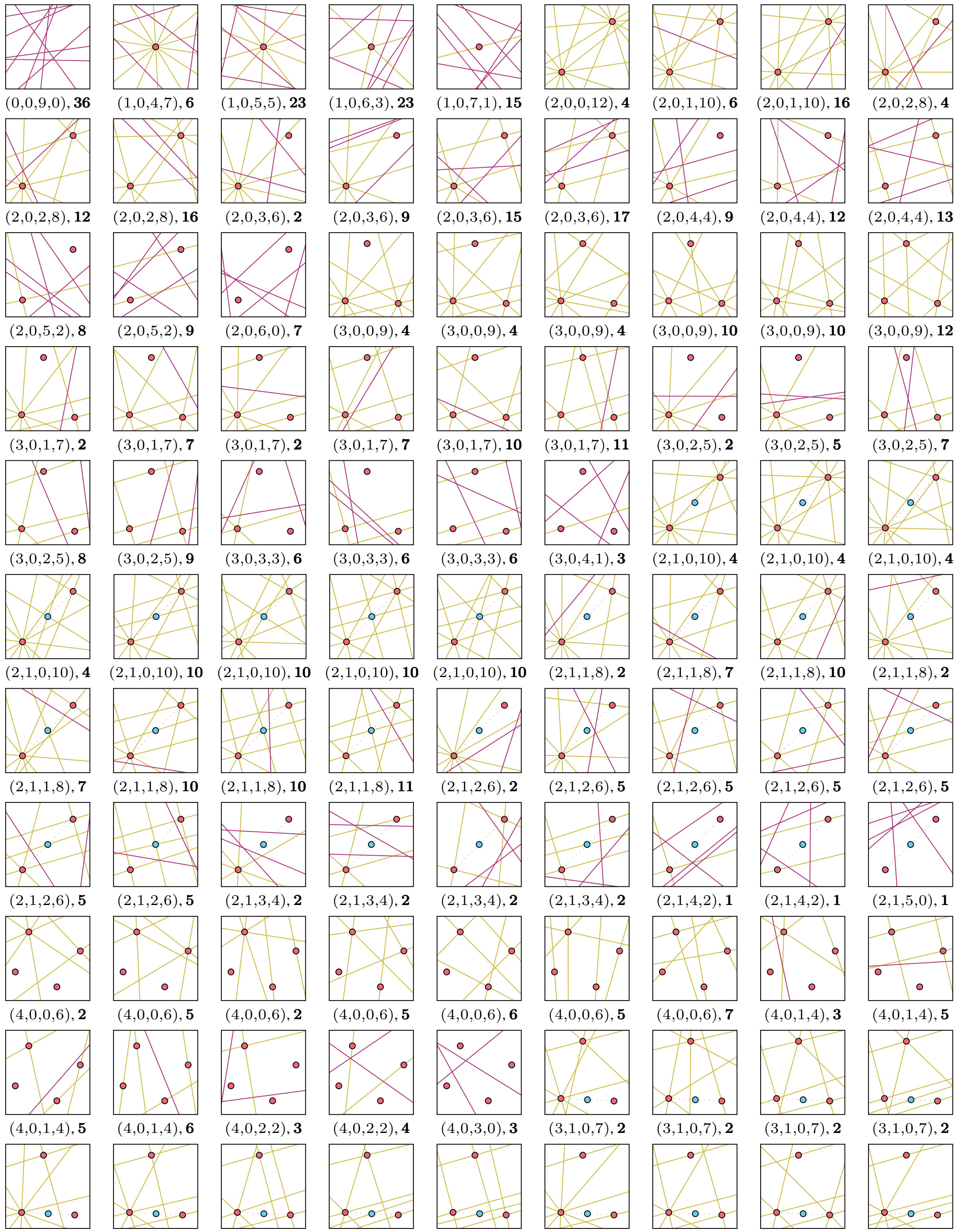
Problem	Arrangement	Camera	Problem	Arrangement	Camera
 $(0,0,2,0), \mathbf{1}$	$\{\overline{e_1e_2}\} \times \{\overline{e_3e_4}\}$	$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$	 $(1,0,0,3), \mathbf{1}$	$\{e_1\} \times \{\overline{e_1e_2}\} \times \{\overline{e_1e_3}\} \times \{\overline{e_1e_4}\}$	$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_5 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
 $(1,0,1,1), \mathbf{1}$	$\{e_1\} \times \{\overline{e_1e_2}\} \times \{\overline{e_3e_4}\}$	$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_5 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$	 $(2,0,0,2), \mathbf{1}$	$\{e_1\} \times \{e_2\} \times \{\overline{e_1e_3}\} \times \{\overline{e_2e_4}\}$	$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_5 & c_6 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
 $(2,0,0,2), \mathbf{1}$	$\{e_1\} \times \{e_2\} \times \{\overline{e_1e_3}\} \times \{\overline{e_1e_4}\}$	$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_5 & c_6 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$	 $(2,0,1,0), \mathbf{1}$	$\{e_1\} \times \{e_2\} \times \{\overline{e_3e_4}\}$	$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_5 & c_6 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
 $(3,0,0,1), \mathbf{1}$	$\{e_1\} \times \{e_2\} \times \{e_3\} \times \{\overline{e_1e_4}\}$	$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_5 & c_6 & c_7 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$	 $(4,0,0,0), \mathbf{1}$	$\{e_1\} \times \{e_2\} \times \{e_3\} \times \{e_4\}$	$\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_5 & c_6 & c_7 & c_8 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Table 8. List of normalizations for single-view PLPs



m	(p^f, p^d, l^f, l^a), algebraic degree								
3	(0,0,9,0), **36**	(1,0,4,7), **6**	(1,0,5,5), **23**	(1,0,6,3), **23**	(1,0,7,1), **15**	(2,0,0,12), **4**	(2,0,1,10), **6**	(2,0,1,10), **16**	(2,0,2,8), **4**
	(2,0,2,8), **12**	(2,0,2,8), **16**	(2,0,3,6), **2**	(2,0,3,6), **9**	(2,0,3,6), **15**	(2,0,3,6), **17**	(2,0,4,4), **9**	(2,0,4,4), **12**	(2,0,4,4), **13**
	(2,0,5,2), **8**	(2,0,5,2), **9**	(2,0,6,0), **7**	(3,0,0,9), **4**	(3,0,0,9), **4**	(3,0,0,9), **4**	(3,0,0,9), **10**	(3,0,0,9), **10**	(3,0,0,9), **12**
	(3,0,1,7), **2**	(3,0,1,7), **7**	(3,0,1,7), **2**	(3,0,1,7), **7**	(3,0,1,7), **10**	(3,0,1,7), **11**	(3,0,2,5), **2**	(3,0,2,5), **5**	(3,0,2,5), **7**
	(3,0,2,5), **8**	(3,0,2,5), **9**	(3,0,3,3), **6**	(3,0,3,3), **6**	(3,0,3,3), **6**	(3,0,4,1), **3**	(2,1,0,10), **4**	(2,1,0,10), **4**	(2,1,0,10), **4**
	(2,1,0,10), **4**	(2,1,0,10), **10**	(2,1,0,10), **10**	(2,1,0,10), **10**	(2,1,0,10), **10**	(2,1,1,8), **2**	(2,1,1,8), **7**	(2,1,1,8), **10**	(2,1,1,8), **2**
	(2,1,1,8), **7**	(2,1,1,8), **10**	(2,1,1,8), **10**	(2,1,1,8), **11**	(2,1,2,6), **2**	(2,1,2,6), **5**	(2,1,2,6), **5**	(2,1,2,6), **5**	(2,1,2,6), **5**
	(2,1,2,6), **5**	(2,1,2,6), **5**	(2,1,3,4), **2**	(2,1,3,4), **2**	(2,1,3,4), **2**	(2,1,3,4), **2**	(2,1,4,2), **1**	(2,1,4,2), **1**	(2,1,5,0), **1**
	(4,0,0,6), **2**	(4,0,0,6), **5**	(4,0,0,6), **2**	(4,0,0,6), **5**	(4,0,0,6), **6**	(4,0,0,6), **5**	(4,0,0,6), **7**	(4,0,1,4), **3**	(4,0,1,4), **5**
	(4,0,1,4), **5**	(4,0,1,4), **6**	(4,0,2,2), **3**	(4,0,2,2), **4**	(4,0,3,0), **3**	(3,1,0,7), **2**	(3,1,0,7), **2**	(3,1,0,7), **2**	(3,1,0,7), **2**
	(3,1,0,7), **2**	(3,1,0,7), **5**	(3,1,0,7), **5**	(3,1,0,7), **5**	(3,1,0,7), **5**	(3,1,0,7), **2**	(3,1,0,7), **5**	(3,1,0,7), **6**	(3,1,0,7), **5**

Table 9. Minimal problems with their associated degree.

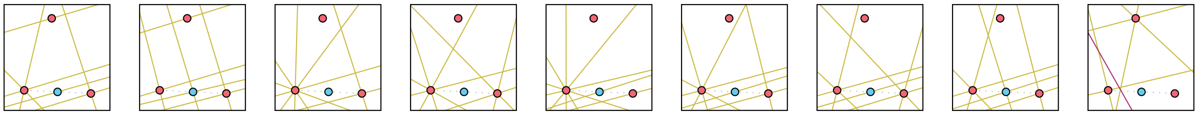
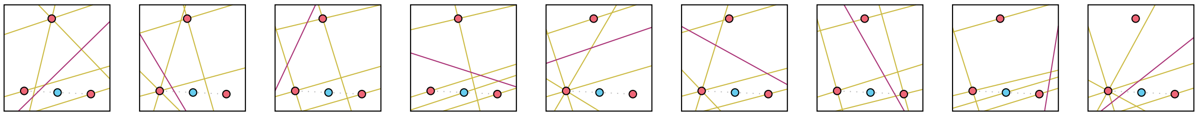
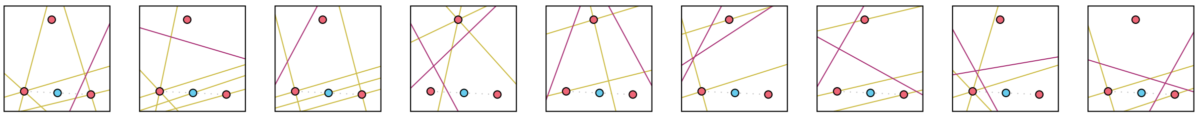
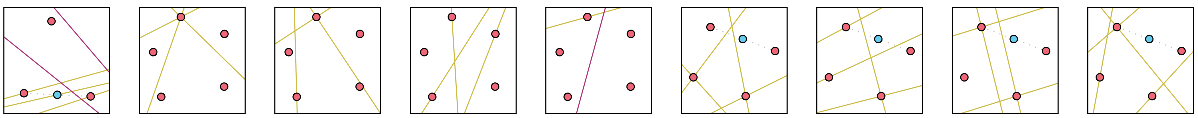
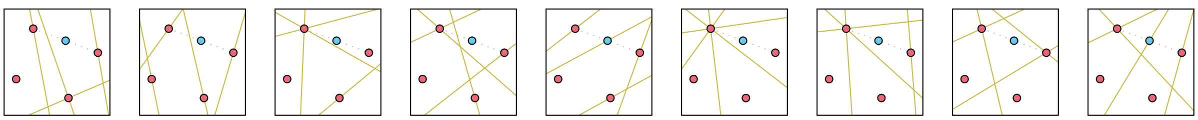
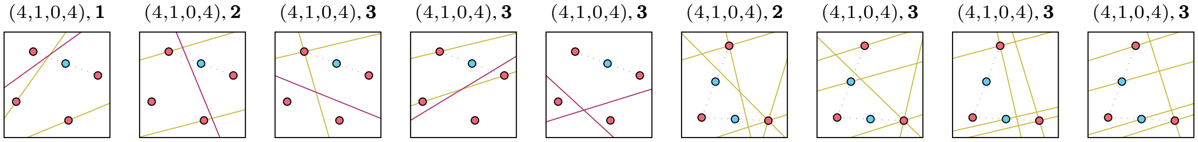
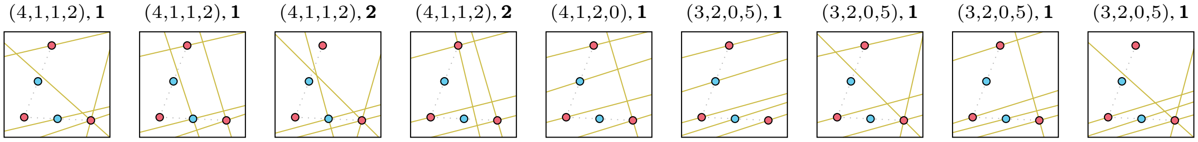
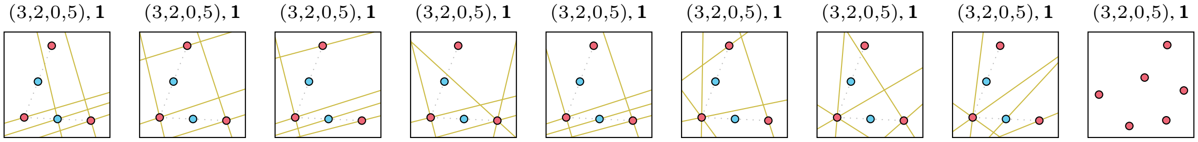
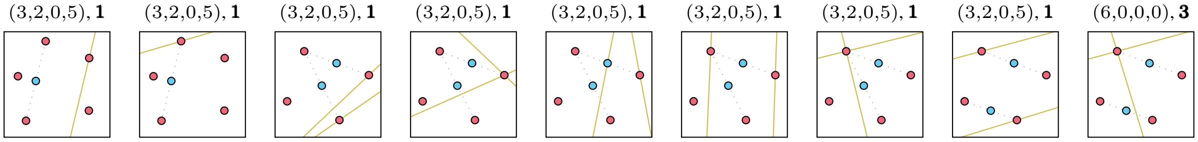
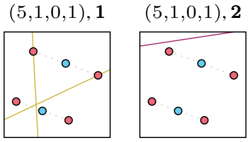
m	(p^f, p^d, l^f, l^a), algebraic degree								
3									
	(3,1,0,7), **6**	(3,1,0,7), **6**	(3,1,0,7), **2**	(3,1,0,7), **5**	(3,1,0,7), **2**	(3,1,0,7), **5**	(3,1,0,7), **5**	(3,1,0,7), **5**	(3,1,1,5), **1**
									
	(3,1,1,5), **1**	(3,1,1,5), **2**	(3,1,1,5), **2**	(3,1,1,5), **2**	(3,1,1,5), **3**	(3,1,1,5), **3**	(3,1,1,5), **3**	(3,1,1,5), **3**	(3,1,1,5), **3**
									
	(3,1,1,5), **4**	(3,1,1,5), **4**	(3,1,1,5), **4**	(3,1,2,3), **1**	(3,1,2,3), **1**	(3,1,2,3), **1**	(3,1,2,3), **1**	(3,1,2,3), **1**	(3,1,2,3), **1**
									
	(3,1,2,3), **1**	(5,0,0,3), **2**	(5,0,0,3), **3**	(5,0,0,3), **4**	(5,0,1,1), **3**	(4,1,0,4), **1**	(4,1,0,4), **1**	(4,1,0,4), **1**	(4,1,0,4), **2**
									
	(4,1,0,4), **1**	(4,1,0,4), **2**	(4,1,0,4), **3**	(4,1,0,4), **3**	(4,1,0,4), **3**	(4,1,0,4), **2**	(4,1,0,4), **3**	(4,1,0,4), **3**	(4,1,0,4), **3**
									
	(4,1,1,2), **1**	(4,1,1,2), **1**	(4,1,1,2), **2**	(4,1,1,2), **2**	(4,1,2,0), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**
									
	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**
									
	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(3,2,0,5), **1**	(6,0,0,0), **3**
									
	(5,1,0,1), **1**	(5,1,0,1), **2**	(4,2,0,2), **1**	(4,2,0,2), **1**	(4,2,0,2), **1**	(4,2,0,2), **1**	(4,2,0,2), **1**	(4,2,0,2), **2**	(4,2,0,2), **1**
									
	(4,2,0,2), **1**	(4,2,1,0), **1**							

Table 10. Minimal problems with their associated degree.

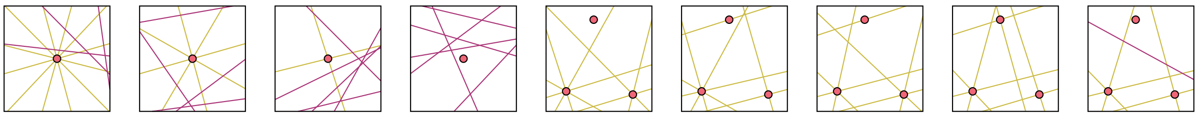
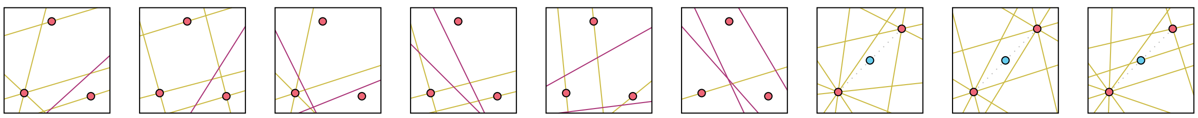
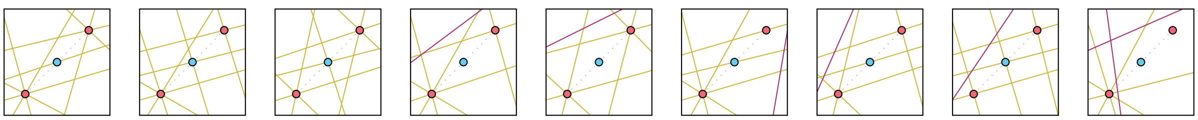
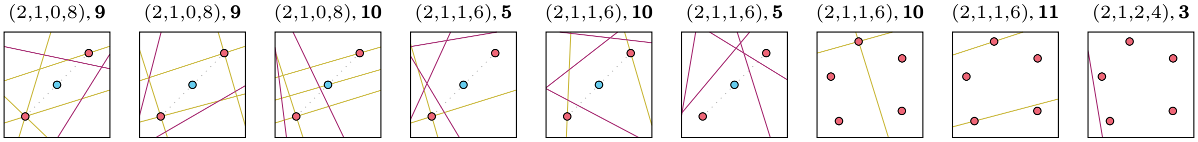
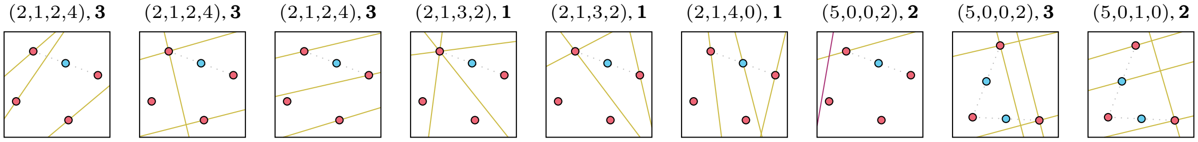
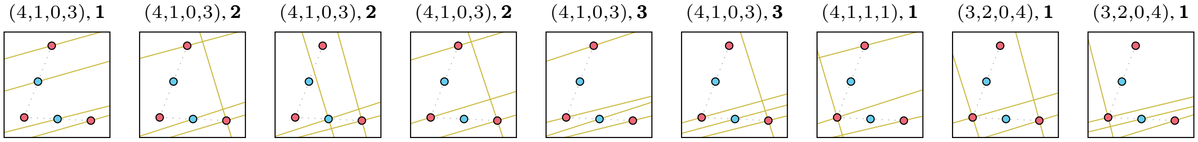
m	(p^f, p^d, l^f, l^a), algebraic degree								
4									
	(1,0,3,6), **2**	(1,0,4,4), **25**	(1,0,5,2), **30**	(1,0,6,0), **12**	(3,0,0,7), **2**	(3,0,0,7), **2**	(3,0,0,7), **8**	(3,0,0,7), **10**	(3,0,1,5), **5**
									
	(3,0,1,5), **6**	(3,0,1,5), **10**	(3,0,2,3), **4**	(3,0,2,3), **6**	(3,0,2,3), **7**	(3,0,3,1), **3**	(2,1,0,8), **2**	(2,1,0,8), **9**	(2,1,0,8), **2**
									
	(2,1,0,8), **9**	(2,1,0,8), **9**	(2,1,0,8), **10**	(2,1,1,6), **5**	(2,1,1,6), **10**	(2,1,1,6), **5**	(2,1,1,6), **10**	(2,1,1,6), **11**	(2,1,2,4), **3**
									
	(2,1,2,4), **3**	(2,1,2,4), **3**	(2,1,2,4), **3**	(2,1,3,2), **1**	(2,1,3,2), **1**	(2,1,4,0), **1**	(5,0,0,2), **2**	(5,0,0,2), **3**	(5,0,1,0), **2**
									
	(4,1,0,3), **1**	(4,1,0,3), **2**	(4,1,0,3), **2**	(4,1,0,3), **2**	(4,1,0,3), **3**	(4,1,0,3), **3**	(4,1,1,1), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**
									
	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**	(3,2,0,4), **1**

Table 11. Minimal problems with their associated degree.

m	(p^f, p^d, l^f, l^a), algebraic degree
5	(1,0,3,5), **6**; (1,0,4,3), **35**; (1,0,5,1), **20**; (4,0,0,4), **3**; (4,0,0,4), **4**; (4,0,0,4), **7**; (4,0,1,2), **3**; (4,0,2,0), **2**; (3,1,0,5), **2**
	(3,1,0,5), **2**; (3,1,0,5), **2**; (3,1,0,5), **4**; (3,1,0,5), **6**; (3,1,0,5), **6**; (3,1,0,5), **4**; (3,1,0,5), **4**; (3,1,0,5), **5**; (3,1,1,3), **1**
	(3,1,1,3), **1**; (3,1,1,3), **2**; (3,1,1,3), **2**
6	(3,0,0,6), **3**; (3,0,0,6), **5**; (3,0,0,6), **12**; (3,0,1,4), **5**; (3,0,1,4), **8**; (3,0,2,2), **3**; (3,0,2,2), **4**; (2,1,0,7), **5**; (2,1,0,7), **5**
	(2,1,0,7), **10**; (2,1,0,7), **10**; (2,1,1,5), **7**; (2,1,1,5), **7**; (2,1,1,5), **10**; (2,1,2,3), **1**; (2,1,2,3), **1**; (2,1,2,3), **1**
7	(2,0,0,8), **3**; (2,0,1,6), **10**; (2,0,2,4), **9**; (2,0,2,4), **20**; (2,0,3,2), **6**; (2,0,3,2), **9**; (2,0,4,0), **3**
8	(1,0,3,4), **10**; (1,0,4,2), **38**; (1,0,5,0), **8**
9	(0,0,6,0), **114**

Table 12. Minimal problems with their associated degree.