WIPES: Wavelet-based Visual Primitives

Supplementary Material

Outline

In this supplementary material, we provide a proof of the sufficiency of wavelet primitives as reconstruction kernels, along with a detailed explanation of their coordinate transformation process, and additional results that could not be included in the main text due to space constraints. The contents are organized as follows:

- **Sec.** A: Proof of the Sufficient Conditions for Using Wavelet Primitives as Reconstruction Kernels, Including Affine Closure, Local Support, and Integral Closure.
- **Sec.** B: The coordinate transformation process of 3D wavelet primitives and their corresponding parameters.
- **Sec.** C: Supplementary experiment for further validates the frequency flexibility of wavelet primitives.

A. Proof of Wavelet Primitives as Reconstruction Kernels

As stated in the EWA splatting framework, a reconstruction kernel must satisfy both *affine closure* and *local support* properties, and its integrated 2D distribution obtained by integrating along one coordinate axis must be computed *explicitly*. In this section, we provide proofs for the affine closure and local support properties of the wavelet primitives and derive their corresponding integrated 2D distribution.

A.1. Proof of Affine Closure for Wavelet Primitives

To demonstrate the affine closure property of the wavelet primitive, we focus on its behavior under an affine transformation $\vec{x}' = \mathbf{A}\vec{x} + \vec{b}$, where $\mathbf{A}$ is an invertible matrix and $\vec{b}$ is a translation vector. We aim to show that there exist transformed parameters $\vec{\mu}'$, $\vec{f}'$, and $\mathbf{\Sigma}'$ such that:

$$\mathcal{W}'(\vec{x}', \vec{\mu}', \vec{f}', \mathbf{\Sigma}') = \frac{1}{2}\left[\cos\left(\vec{f}' \cdot (\vec{x}' - \vec{\mu}')\right) + 1\right] \mathcal{G}(\vec{x}', \vec{\mu}', \mathbf{\Sigma}'). \quad (1)$$

Step 1: Affine Invariance of the Gaussian Term

The Gaussian component $\mathcal{G}(\vec{x}, \vec{\mu}, \mathbf{\Sigma})$ inherently satisfies affine invariance. Under the transformation $\vec{x}' = \mathbf{A}\vec{x} + \vec{b}$, the parameters of the Gaussian update as:

$$\begin{aligned} \vec{\mu}' &= \mathbf{A}\vec{\mu} + \vec{b}, \\ \mathbf{\Sigma}' &= \mathbf{A}\mathbf{\Sigma}\mathbf{A}^T. \end{aligned} \quad (2)$$

This ensures that $\mathcal{G}(\vec{x}, \vec{\mu}, \mathbf{\Sigma})$ retains its functional form in the transformed coordinates:

$$\mathcal{G}(\vec{x}, \vec{\mu}, \mathbf{\Sigma}) \propto \mathcal{G}(\vec{x}', \vec{\mu}', \mathbf{\Sigma}'), \quad (3)$$

where the proportionality constant arises from normalization factors but does not affect the closure property.

Step 2: Analyze the cosine modulation term

The critical step lies in proving that the cosine term $\cos\left(\vec{f} \cdot (\vec{x} - \vec{\mu})\right)$ preserves its structure under the affine transformation. Substituting $\vec{x} = \mathbf{A}^{-1}(\vec{x}' - \vec{b})$ into the original inner product:

$$\vec{f} \cdot (\vec{x} - \vec{\mu}) = \vec{f}^T \left(\mathbf{A}^{-1}(\vec{x}' - \vec{b}) - \vec{\mu}\right). \quad (4)$$

Rearranging terms:

$$\vec{f}^\top \mathbf{A}^{-1}(\vec{x}' - \mathbf{A}\vec{\mu} - \vec{b}) = \vec{f}^\top \mathbf{A}^{-1}(\vec{x}' - \vec{\mu}'). \quad (5)$$

To maintain the inner product form $\vec{f}' \cdot (\vec{x}' - \vec{\mu}')$, we define the transformed frequency vector as:

$$\vec{f}' = \mathbf{A}^{-\top}\vec{f}, \quad (6)$$

where $\mathbf{A}^{-\top} = (\mathbf{A}^{-1})^\top$. This ensures:

$$\vec{f}' \cdot (\vec{x}' - \vec{\mu}') = \left(\mathbf{A}^{-\top}\vec{f}\right)^\top (\vec{x}' - \vec{\mu}') = \vec{f}^\top \mathbf{A}^{-1}(\vec{x}' - \vec{\mu}'), \quad (7)$$

which matches the original expression. Thus, the cosine modulation term retains its functional form under the affine transformation.

Step 3: Parameter Transformation and Closure

Combining the above results, for any affine transformation $\vec{x}' = \mathbf{A}\vec{x} + \vec{b}$, the function $\mathcal{W}$ remains closed by updating its parameters as:

$$\vec{\mu}' = \mathbf{A}\vec{\mu} + \vec{b}, \quad \mathbf{\Sigma}' = \mathbf{A}\mathbf{\Sigma}\mathbf{A}^T, \quad \vec{f}' = \mathbf{A}^{-\top}\vec{f}. \quad (8)$$

The transformed function becomes:

$$\mathcal{W}'(\vec{x}', \vec{\mu}', \vec{f}', \mathbf{\Sigma}') = \frac{1}{2}\left[\cos\left(\vec{f}' \cdot (\vec{x}' - \vec{\mu}')\right) + 1\right] \mathcal{G}(\vec{x}', \vec{\mu}', \mathbf{\Sigma}'), \quad (9)$$

A.2. Proof of Local Support for Wavelet Primitives

The wavelet primitives exhibits local support due to the following two key properties:

Step 1: Dominant Exponential Decay from the Gaussian Kernel

The Gaussian term $\mathcal{G}(\vec{x}, \vec{\mu}, \mathbf{\Sigma})$ decays exponentially as $\|\vec{x} - \vec{\mu}\| \to \infty$:

$$\mathcal{G} \sim \exp\left(-\frac{1}{2}(\vec{x} - \vec{\mu})^T \mathbf{\Sigma}^{-1} (\vec{x} - \vec{\mu})\right). \tag{10}$$

For any positive-definite $\mathbf{\Sigma}$, the quadratic form $(\vec{x} - \vec{\mu})^T \mathbf{\Sigma}^{-1} (\vec{x} - \vec{\mu})$ grows quadratically with $\|\vec{x} - \vec{\mu}\|$, ensuring $\mathcal{G}$ vanishes rapidly outside a neighborhood of $\vec{\mu}$.

Step 2: Bounded Modulating Cosine Term

The cosine modulation term satisfies:

$$0 \leq \frac{1}{2}\left[\cos\left(\vec{f} \cdot (\vec{x} - \vec{\mu})\right) + 1\right] \leq 1 \quad \forall \vec{x}. \tag{11}$$

While this term introduces spatial oscillations, its bounded amplitude does not counteract the exponential decay of $\mathcal{G}$.

Step 3: Final Decay Behavior of Wavelet Primitives

Combining these results:

$$|\mathcal{W}(\vec{x})| \leq \mathcal{G}(\vec{x}, \vec{\mu}, \mathbf{\Sigma}), \tag{12}$$

which inherits the Gaussian's exponential decay. Thus, $\mathcal{W}(\vec{x})$ vanishes rapidly as $\|\vec{x} - \vec{\mu}\|$ increases, confirming its strict local support around $\vec{\mu}$.

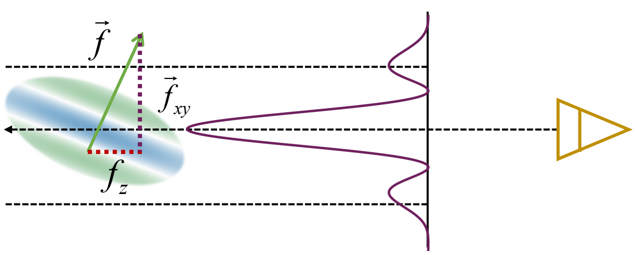

Figure 1. Projecting wavelet primitives along the z-axis onto the image plane.

A.3. Proof of the Closed-Form Integration for Wavelet Primitives

Given the 3d wavelet primitives defined in Equation 6 in the main text, the integration along the z-axis is calculated as follows:

$$\int_{\mathbb{R}} \mathcal{W}(\vec{x}, \vec{\mu}, \vec{f}, \mathbf{\Sigma}) dz = \int_{\mathbb{R}} \frac{1}{2}[\cos(\vec{f} \cdot (\vec{x} - \vec{\mu})) + 1]\, \mathcal{G}(\vec{x}, \vec{\mu}, \mathbf{\Sigma}) dz, \tag{13}$$

We decompose the above integral into two parts, denoted as I_1 and I_2:

$$\begin{aligned} I_1 &= \int_{\mathbb{R}} \frac{1}{2}\cos(\vec{f} \cdot (\vec{x} - \vec{\mu}))\mathcal{G}(\vec{x}, \vec{\mu}, \mathbf{\Sigma}) dz \\ I_2 &= \int_{\mathbb{R}} \frac{1}{2}\mathcal{G}(\vec{x}, \vec{\mu}, \mathbf{\Sigma}) dz. \end{aligned} \tag{14}$$

Since the integral result of I_2 can be directly obtained by applying Equation 2 in the main text:

$$I_2 = \frac{1}{2}\mathcal{G}'(\vec{x}', \vec{\mu}', \mathbf{\Sigma}'), \tag{15}$$

we provide a detailed derivation of the integral I_1 in the following section.

Step 1: Expanding the Cosine Term

By applying Euler's formula $e^{ix} = \cos x + i \sin x$, we expand the cosine term in I_1 as follows:

$$I_1 = \frac{1}{2}\int_{\mathbb{R}} \mathrm{Re}\left[e^{i\vec{f}\cdot(\vec{x}-\vec{\mu})}\mathcal{G}(\vec{x}, \vec{\mu}, \mathbf{\Sigma})\right] dz. \tag{16}$$

Leveraging the linearity of integration and the linear structure of complex numbers, the integral simplifies to:

$$I_1 = \frac{1}{2}\mathrm{Re}\left[\int_{\mathbb{R}} e^{i\vec{f}\cdot(\vec{x}-\vec{\mu})}\mathcal{G}(\vec{x}, \vec{\mu}, \mathbf{\Sigma}) dz\right]. \tag{17}$$

Step 2: Variable and Matrix Decomposition

We decompose the vectors $\vec{x}$, $\vec{\mu}$, and $\vec{f}$ into components parallel and perpendicular to the xy-plane. Specifically, each vector is separated into its xy-plane projection and the corresponding component along the z-axis:

$$\vec{x} = \begin{bmatrix} \vec{x}' \\ z \end{bmatrix}, \quad \vec{\mu} = \begin{bmatrix} \vec{\mu}' \\ \mu_z \end{bmatrix}, \quad \vec{f} = \begin{bmatrix} \vec{f}' \\ f_z \end{bmatrix}, \tag{18}$$

where $\vec{x}' = (x, y)^\top$, $\vec{\mu}' = (\mu_x, \mu_y)^\top$, $\vec{f}' = (f_x, f_y)^\top$.

Similarly, the covariance matrix is partitioned into a block matrix:

$$\mathbf{\Sigma}^{-1} = \begin{bmatrix} \mathbf{A} & \vec{B} \\ \vec{B}^\top & C \end{bmatrix}, \tag{19}$$

where $\mathbf{A} \in \mathbb{R}^{2\times 2}$, $\vec{B} \in \mathbb{R}^{2\times 1}$, and $C \in \mathbb{R}$.

Step 3: Substituting Decomposed Variables

Substituting the decomposed vectors and matrix into the exponential and Gaussian terms, we expand the exponential term as:

$$e^{i\vec{f}\cdot(\vec{x}-\vec{\mu})} = e^{i\vec{f}'\cdot(\vec{x}'-\vec{\mu}')}e^{if_z(z-\mu_z)}. \tag{20}$$

Similarly, the Gaussian term expands to:

$$e^{-\frac{1}{2}(\vec{x}'-\vec{\mu}')^\top \mathbf{A}(\vec{x}'-\vec{\mu}') + (\vec{x}'-\vec{\mu}')^\top \vec{B}(z-\mu_z) - \frac{1}{2}C(z-\mu_z)^2}. \tag{21}$$

Since the integration is with respect to z, we treat all terms containing z as integration variables and regard all remaining terms as constants. Thus, the integral becomes:

$$I_1 = \frac{1}{2}\mathrm{Re}\left[e^{i\vec{f}'\cdot(\vec{x}'-\vec{\mu}')}e^{-\frac{1}{2}(\vec{x}'-\vec{\mu}')^\top \mathbf{A}(\vec{x}'-\vec{\mu}')}I_z\right], \tag{22}$$

where

$$I_z = \int_{\mathbb{R}} e^{if_z(z-\mu_z)}e^{-\frac{1}{2}C(z-\mu_z)^2 - (\vec{x}'-\vec{\mu}')^\top \vec{B}(z-\mu_z)} dz. \tag{23}$$

Then, we combine the exponential terms associated with z:

$$s = -\frac{1}{2}C(z-\mu_z)^2 - (\vec{x}' - \vec{\mu}')^\top \vec{B}(z-\mu_z) + if_z(z-\mu_z). \tag{24}$$

For convenience, we define:

$$v = -(\vec{x}' - \vec{\mu}')^\top \vec{B} + if_z, \tag{25}$$

which allows us to express s as:

$$s = -\frac{1}{2}C(z-\mu_z)^2 + v(z-\mu_z). \tag{26}$$

Step 4: Completing the Square and Evaluating the Integral

We complete the square for the quadratic terms in s as follows:

$$s = -\frac{1}{2}C\left(z-\mu_z-\frac{v}{C}\right)^2 + \frac{v^2}{2C}. \tag{27}$$

Substituting, the integral I_z becomes:

$$I_z = e^{\frac{v^2}{2C}} \int_{\mathbb{R}} e^{-\frac{1}{2}C\left(z-\mu_z-\frac{v}{C}\right)^2} dz. \tag{28}$$

Applying the change of variable $t = z - \mu_z - \frac{v}{C}$ transforms the integral into a standard Gaussian form:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}Ct^2}\,\mathrm{d}t = \sqrt{\frac{2\pi}{C}}. \tag{29}$$

Thus, I_z evaluates to:

$$e^{\frac{v^2}{2C}}\sqrt{\frac{2\pi}{C}}. \tag{30}$$

Step 5: Combining Results and Simplification

Substituting I_z into the expression for I_1, and expanding v, we obtain the following.

$$\begin{aligned} I_1 &= \frac{1}{2}\sqrt{\frac{2\pi}{C}}\, e^{-\frac{1}{2}(\vec{x}'-\vec{\mu}')^\top\left(\mathbf{A}-\frac{\vec{B}\vec{B}^\top}{C}\right)(\vec{x}'-\vec{\mu}')-\frac{f_z^2}{2C}} \\ &\times \cos\left(\left(\vec{f}' + \frac{f_z\vec{B}^\top}{C}\right)\cdot(\vec{x}'-\vec{\mu}')\right). \end{aligned} \tag{31}$$

Figure x illustrates that when projecting wavelet primitives along the z-axis onto the image plane, the frequency component associated with the z-axis vanishes. Similarly, as described in EWA splatting, the third row and third column of the covariance matrix, corresponding to the z-axis, are also set to zero during projection. Therefore, setting $f_z = 0$ and the corresponding elements of vector $\vec{B} = \vec{0}$ in the equation above, we obtain the final integral I_1 as follows:

$$I_1 = \frac{1}{2}\sqrt{2\pi}\,\mathcal{G}'(\vec{x}', \vec{\mu}', \mathbf{\Sigma}')\cos\left(\vec{f}'\cdot(\vec{x}'-\vec{\mu}')\right). \tag{32}$$

Conclusion

Since the constant terms $\sqrt{2\pi}$ in the integral I_1 can be absorbed into the opacity parameter, which is learned during the optimization process, we obtain the following simplified form:

$$I_1 = \frac{1}{2}\mathcal{G}'(\vec{x}', \vec{\mu}', \mathbf{\Sigma}')\cos\left(\vec{f}'\cdot(\vec{x}'-\vec{\mu}')\right). \tag{33}$$

Finally, we obtain the following expression for the 3D wavelet primitive, as shown in Equation 7 in the main text:

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{W}(\vec{x}, \vec{\mu}, \vec{f}, \mathbf{\Sigma})dz &= \frac{1}{2}[\cos\left(\vec{f}'\cdot(\vec{x}'-\vec{\mu}')\right) + 1]\mathcal{G}'(\vec{x}', \vec{\mu}', \mathbf{\Sigma}') \\ &= \mathcal{W}'(\vec{x}', \vec{\mu}', \vec{f}', \mathbf{\Sigma}') \end{aligned} \tag{34}$$

B. Detailed of the Coordinate Transformation for 3D Wavelet Primitives

In this section, we provide a detailed explanation of the coordinate transformation process for 3D wavelet primitives. To better differentiate among the various coordinate spaces, we denote the wavelet primitive in world space as $\mathcal{W}_w$, in camera space as $\mathcal{W}_c$, in ray space as $\mathcal{W}_r$, and in image space as $\mathcal{W}_i$, and other related variables are defined similarly.

First, we use the world-to-camera matrix $\mathbf{W}$ to convert the world space $\vec{x}_w$ into the camera space $\vec{x}_c$

$$\vec{x}_c = \mathbf{W}\vec{x}_w. \tag{35}$$

Then, we substitute $\vec{x}_w = \mathbf{W}^{-1}\vec{x}_c$ back into the wavelet primitive expression in world space, and obtain the frequency vector and covariance matrix of the wavelet primitive in the camera space:

$$\begin{aligned} \vec{f}_c &= \vec{f}_w\mathbf{W}^{-1} \\ \mathbf{\Sigma}_c &= \mathbf{W}\mathbf{\Sigma}_w\mathbf{W}^\top \end{aligned} \tag{36}$$

Secondly, we follow the same approach as EWA splatting and 3DGS to transform the wavelet reconstruction kernel from the camera space into ray space, compute the Jacobian matrix $\mathbf{J}$, and approximate the mapping as an affine transformation using a first-order Taylor expansion:

$$x_r = Jx_c. \tag{37}$$

Similarly, we substitute $\vec{x}_c = \mathbf{J}^{-1}\vec{x}_r$ back into the wavelet primitive expression in the camera space, obtaining the frequency vector and covariance matrix of the wavelet primitive in ray space:

$$\begin{aligned} \vec{f}_r &= \vec{f}_c\mathbf{J}^{-1} \\ \mathbf{\Sigma}_r &= \mathbf{J}\mathbf{\Sigma}_c\mathbf{J}^\top. \end{aligned} \tag{38}$$

In summary, we obtain the relationship between the frequency vector and the covariance matrix in ray space and world space:

$$\begin{aligned}\vec{f}_r &= \vec{f}_w \mathbf{W}^{-1}\mathbf{J}^{-1} \\ &= \vec{f}_w(\mathbf{J}\mathbf{W})^{-1} \\ \mathbf{\Sigma}_r &= \mathbf{J}\mathbf{W}\mathbf{\Sigma}_w\mathbf{J}^\top\mathbf{W}^\top,\end{aligned} \tag{39}$$

which is consistent with the results presented in the main text.

Finally, we project the ray-space wavelet primitive directly into image space. The covariance matrix and frequency vector in image space correspond to the 2×2 submatrix and 2×1 subvector of the ray-space covariance matrix and frequency vector, respectively.

C. Additional Results For the Frequency Flexibility of Wavelet Primitives.

To thoroughly validate that the proposed wavelet primitives effectively represent both high- and low-frequency components of visual signals, rather than being limited to high-frequency features, we conducted experiments using 100k primitives for Gaussian and wavelets primitives. We randomly selected 10 images from the DIV2K dataset and applied Gaussian blur with $\sigma = 3, 7, 11$. Since WIPES adaptively learns the frequency parameters, it consistently outperforms Gaussian primitives across all blur levels (Tab. 1).

Table 1. PSNR and MS-SSIM across blur levels.

Method	level-1 ($\sigma = 3$)		level-2 ($\sigma = 7$)		level-3 ($\sigma = 11$)	
	PSNR	MS-SSIM	PSNR	MS-SSIM	PSNR	MS-SSIM
GS	49.07	0.9991	51.14	**0.9996**	50.64	0.9990
WIPES	**55.45**	**0.9995**	**55.35**	0.9995	**56.48**	**0.9995**