Supplementary Material for the Paper:
Seeing Neural Implicit Functions as Fourier Series

1. Images used in image regression task

Figure 1 shows the images used for the image regression task in the main paper.

2. Proof of equation (7)

We will prove the validity of equation (7) in the general case of dimension $d \in \mathbb{N}$. We use the concept of mathematical induction for this task. Therefore we show, that the equation is true for $d = 1$ and additionally prove, that if the equation holds for dimension $d-1$ it is also valid for dimension d.

$d = 1$:

$$\begin{aligned}
f(x) &= \sum_{n\in\mathbb{Z}^1} c_n e^{2\pi i n\cdot x} \\
&= \sum_{n\in\mathbb{N}} c_n e^{2\pi i n\cdot x} + \sum_{n\in\mathbb{N}} c_{-n} e^{-2\pi i n\cdot x} + c_0 \\
&\overset{c_n^* = c_{-n}}{=} \sum_{n\in\mathbb{N}} \big(\mathrm{Re}(c_n) + i\mathrm{Im}(c_n)\big)\big(\cos(2\pi n x) + i\sin(2\pi n x)\big) \\
&+ \sum_{n\in\mathbb{N}} \big(\mathrm{Re}(c_n) - i\mathrm{Im}(c_n)\big)\big(\cos(2\pi n x) - i\sin(2\pi n x)\big) \\
&+ c_0 \\
&= \sum_{n\in\mathbb{N}} 2\mathrm{Re}(c_n)\cos(2\pi n x) - 2\mathrm{Im}(c_n)\sin(2\pi n x) \\
&+ c_0 \\
&= \sum_{n\in\mathbb{N}_0} a_n\cos(2\pi n x) + b_n \sin(2\pi n x),
\end{aligned} \tag{1}$$

where

$$a_0 = c_0,\ a_n = 2\mathrm{Re}(c_n),\ b_n = -2\mathrm{Im}(c_n). \tag{2}$$

Assumption of the induction:
We will assume that the equation holds for $d-1$, where $d \geq 2$.

Induction step: $d-1 \to d$:
As the fourier series of any periodic and continous function is absolutely convergent, we are allowed to rearrange the sum in $(*)$ and receive

$$\begin{aligned}
&\sum_{\mathbf{n}=(n_1,\dots,n_d)\in\mathbb{Z}^d} c_{\mathbf{n}} e^{2\pi i \mathbf{n}\cdot\mathbf{x}} \\
&\overset{(*)}{=} \sum_{n_1\in\mathbb{N}} \sum_{(n_2,\dots,n_d)\in\mathbb{Z}^{d-1}} c_{\mathbf{n}} e^{2\pi i \mathbf{n}\cdot\mathbf{x}} \\
&+ \sum_{n_1\in\mathbb{N}} \sum_{(n_2,\dots,n_d)\in\mathbb{Z}^{d-1}} c_{-\mathbf{n}} e^{-2\pi i \mathbf{n}\cdot\mathbf{x}} \\
&+ \sum_{n_1=0}^{0} \sum_{(n_2,\dots,n_d)\in\mathbb{Z}^{d-1}} c_{\mathbf{n}} e^{2\pi i \mathbf{n}\cdot\mathbf{x}} \\
&\overset{c_{\mathbf{n}}^* = c_{-\mathbf{n}}}{=} \sum_{\mathbf{n}\in\mathbb{N}\times\mathbb{Z}^{d-1}} 2\mathrm{Re}(c_{\mathbf{n}})\cos(2\pi\mathbf{n}\cdot\mathbf{x}) - 2\mathrm{Im}(c_{\mathbf{n}})\sin(2\pi\mathbf{n}\cdot\mathbf{x}) \\
&+ \sum_{\mathbf{n}\in\{0\}\times\mathbb{Z}^{d-1}} c_{\mathbf{n}} e^{2\pi i \mathbf{n}\cdot\mathbf{x}} \\
&\overset{\text{Ind. asm.}}{=} \sum_{\mathbf{n}\in\mathbb{N}\times\mathbb{Z}^{d-1}} 2\mathrm{Re}(c_{\mathbf{n}})\cos(2\pi\mathbf{n}\cdot\mathbf{x}) - 2\mathrm{Im}(c_{\mathbf{n}})\sin(2\pi\mathbf{n}\cdot\mathbf{x}) \\
&+ \sum_{\mathbf{n}\in\{0\}\times\mathbb{N}_0\times\mathbb{Z}^{d-2}} a'_{\mathbf{n}}\cos(2\pi\mathbf{n}\cdot\mathbf{x}) + b'_{\mathbf{n}}\sin(2\pi\mathbf{n}\cdot\mathbf{x}),
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
a'_{\mathbf{0}} &= c_{\mathbf{0}}, \\
a'_{\mathbf{n}} &= \begin{cases} 0 & \exists j \in \{3,\dots,d\} : n_2 = \dots = n_{j-1} = 0 \wedge n_j < 0 \\ 2\mathrm{Re}(c_{\mathbf{n}}) & \text{otherwise,} \end{cases} \\
b'_{\mathbf{n}} &= \begin{cases} 0 & \exists j \in \{3,\dots,d\} : n_2 = \dots = n_{j-1} = 0 \wedge n_j < 0 \\ -2\mathrm{Im}(c_{\mathbf{n}}) & \text{otherwise.} \end{cases}
\end{aligned} \tag{4}$$

Combining these two summands we get

$$\sum_{\mathbf{n}\in\mathbb{N}_0\times\mathbb{Z}^{d-1}} a_{\mathbf{n}}\cos(2\pi\mathbf{n}\cdot\mathbf{x}) + b_{\mathbf{n}}\sin(2\pi\mathbf{n}\cdot\mathbf{x}), \tag{5}$$

(a) Elephants (b) Spaceship (c) Fox (d) Pyramid (e) Plants

(f) Islamic arch (g) Pattern (h) Football (i) Butterfly (j) COVID-19

Figure 1: The images used in the image regression experiments.

where

$$
\begin{aligned}
a_{\mathbf{0}} &= c_{\mathbf{0}}, \\
a_{\mathbf{n}} &= \begin{cases} 0 & \exists j \in \{2, \dots, d\} : n_1 = \dots = n_{j-1} = 0 \wedge n_j < 0 \\ 2\mathrm{Re}(c_{\mathbf{n}}) & \text{otherwise,} \end{cases} \\
b_{\mathbf{n}} &= \begin{cases} 0 & \exists j \in \{2, \dots, d\} : n_1 = \dots = n_{j-1} = 0 \wedge n_j < 0 \\ -2\mathrm{Im}(c_{\mathbf{n}}) & \text{otherwise.} \end{cases}
\end{aligned} \tag{6}
$$

3. Proof of equation (11)

In the following we use $|\cdot|$ to talk about the number of elements in a set. Furthermore, we use the notation $[\![n]\!] := \{0, \dots, n\}$ for $n \in \mathbb{N}$ and $[\![m, l]\!] := \{m, \dots, l\}$ for $m, l \in \mathbb{Z}$ and $m < l$. We have

$$
\begin{aligned}
\mathbf{B} = \{0, \dots, N\} \times \{-N, \dots, N\}^{d-1} \setminus \{\mathbf{n} \in \mathbb{N}_0 \times \mathbb{Z}^{d-1} : \\
\exists j \in \{2, \dots, d\} : n_1 = \dots = n_{j-1} = 0 \wedge n_j < 0\}.
\end{aligned}
$$

It is immediately clear, that

$$
|\{0, \dots, N\} \times \{-N, \dots, N\}^{d-1}| = (N+1)(2N+1)^{d-1},
$$

therefore the only thing we need to show is, that

$$
\begin{aligned}
|\{\mathbf{n} \in [\![N]\!] \times [\![-N, N]\!]^{d-1} : \exists j \in \{2, \dots, d\} : \\
n_1 = \dots = n_{j-1} = 0 \wedge n_j < 0\}| \\
= \sum_{l=0}^{d-2} N(2N+1)^l.
\end{aligned}
$$

We will do this proof with mathematical induction. We start with $d = 2$:

$$
\begin{aligned}
&|\{\mathbf{n} \in [\![N]\!] \times [\![-N, N]\!] : \exists j \in \{2\} : n_1 = 0 \wedge n_j < 0\}| \\
&= |\{\mathbf{n} \in \{0\} \times [\![-N, -1]\!]\}| \\
&= N
\end{aligned}
$$

Assumption of the induction:
We will assume that the equation holds for some d, where $d \geq 2$.

Induction step: $d \to d+1$:

$$
\begin{aligned}
&|\{\mathbf{n} \in [\![N]\!] \times [\![-N, N]\!]^d : \exists j \in [\![2, d+1]\!] : \\
&\quad n_1 = \dots = n_{j-1} = 0 \wedge n_j < 0\}| \\
&= |\{\mathbf{n} \in [\![N]\!] \times [\![-N, N]\!]^d : \exists j \in [\![3, d+1]\!] : \\
&\quad n_1 = \dots = n_{j-1} = 0 \wedge n_j < 0\}| + \\
&\quad |\{\mathbf{n} \in [\![N]\!] \times [\![-N, N]\!]^d : \exists j \in \{2\} : \\
&\quad n_1 = \dots = n_{j-1} = 0 \wedge n_j < 0\}| \\
&= |\{\mathbf{n} \in [\![N]\!] \times [\![-N, N]\!]^d : \exists j \in [\![3, d+1]\!] : \\
&\quad n_1 = \dots = n_{j-1} = 0 \wedge n_j < 0\}| + \\
&\quad |\{\mathbf{n} \in \{0\} \times [\![-N, -1]\!] \times [\![-N, N]\!]^{d-1}\}| \\
&= |\{\mathbf{n} \in [\![N]\!] \times [\![-N, N]\!]^{d-1} : \exists j \in [\![2, d]\!] : \\
&\quad n_1 = \dots = n_{j-1} = 0 \wedge n_j < 0\}| + \\
&\quad N(2N+1)^{d-1} \\
&\overset{\text{Ind. asm.}}{=} \sum_{l=0}^{d-2} N(2N+1)^l + N(2N+1)^{d-1} = \sum_{l=0}^{d-1} N(2N+1)^l.
\end{aligned}
$$

4. Proof of equation (12)

Combining equation (2) and (3) we get

$$\mathbf{y}(\mathbf{x}, \mathbf{W}) = \mathbf{W} \cdot \begin{pmatrix} cos(2\pi \mathbf{B} \cdot \mathbf{x}) \\ sin(2\pi \mathbf{B} \cdot \mathbf{x}) \end{pmatrix} + \mathbf{b}.$$

If we set $\mathbf{B} = (\mathbf{B}_1, \dots, \mathbf{B}_m)^T$, with $\mathbf{B}_i \in \mathbb{R}^{1\times d}$, then the first summand is equal to

$$\begin{pmatrix} \sum_{k=1}^m W_{1,k} c(2\pi \mathbf{B}_k x) + \sum_{k=1}^m W_{1,m+k} s(2\pi \mathbf{B}_k x) \\ \vdots \\ \sum_{k=1}^m W_{d_o,k} c(2\pi \mathbf{B}_k x) + \sum_{k=1}^m W_{d_o,m+k} s(2\pi \mathbf{B}_k x) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=1}^m W_{1,k} s(2\pi \mathbf{B}_k x - \pi/2) + \sum_{k=1}^m W_{1,m+k} s(2\pi \mathbf{B}_k x) \\ \vdots \\ \sum_{k=1}^m W_{d_o,k} s(2\pi \mathbf{B}_k x - \pi/2) + \sum_{k=1}^m W_{d_o,m+k} s(2\pi \mathbf{B}_k x), \end{pmatrix}$$

where s and c are short forms of sine and cosine. And if we define $\phi = (-\pi/2, \dots, -\pi/2, 0, \dots, 0)^T \in \mathbb{R}^{2m}$ and $\mathbf{C} := (\mathbf{B}, \mathbf{B})^T$, we result in

$$\mathbf{y}(\mathbf{x}, \mathbf{W}) = \mathbf{W} \cdot sin(2\pi \mathbf{C} \cdot \mathbf{x} + \phi)^T + \mathbf{b}.$$

5. Periodicity in MLP

We claim that when an integer mapping is applied to the input, the network output is forced to be periodic. this comes from the fact that the frequencies introduced by the activations are integers and a periodic signal has only integer frequencies. To prove this claim, we will first analyze the frequencies in the 1D case and later demonstrate those findings in 2D experiments. As an initial Fourier mapping involves the usage of a sinus function on the mapped input, we discuss now the effect of applying an activation function on top of a sinus representation. Applying a ReLU or Sine on a mapped input, will produce frequencies that are multiples of its input frequencies. For example, if we apply a ReLU to a Sine function we get

$$ReLU(\sin(x)) = \frac{1}{\pi} + \frac{\sin(x)}{2} + \sum_{\substack{n=2k \\ k\in\mathbb{N}}} \frac{2}{\pi(1-n^2)} \cos(nx), \quad (7)$$

and if we apply a Sine to a Sine we get

$$\sin(A \cdot \sin(x)) = 2 \sum_{n=0}^{\infty} J_{2n+1}(A) \sin((2n+1)x), \quad (8)$$

where J_i are Bessel functions. In these cases we can immediately see that, the output frequencies are multiples of the input frequencies. Motivated by these findings we explore the question, whether it does generalize to higher dimensional signals.

To do so, we define $\mathbf{B}$ in two different ways. First we generate $\mathbf{B}_N$ limited by $N = 2$, responsible for the integer mapping and for the Gauss mapping we sample $\mathbf{B}$ from a Gaussian distribution with mean, variance and dimension according to the previous $\mathbf{B}_N$, to achieve maximal comparability. We then compare the spectrum of $f_1 := \gamma(x) \cdot \mathbf{1}$, $f_2 := \text{ReLu}(\gamma(x) \cdot \mathbf{1})$ and $f_3 := \sin(\gamma(x) \cdot \mathbf{1})$, where $\mathbf{1}$ represents the weight matrix, in this example defined to only contain 1's. We visualize the spectrum of our results in Fig. 2 in the range of $(0, 10)^2$.

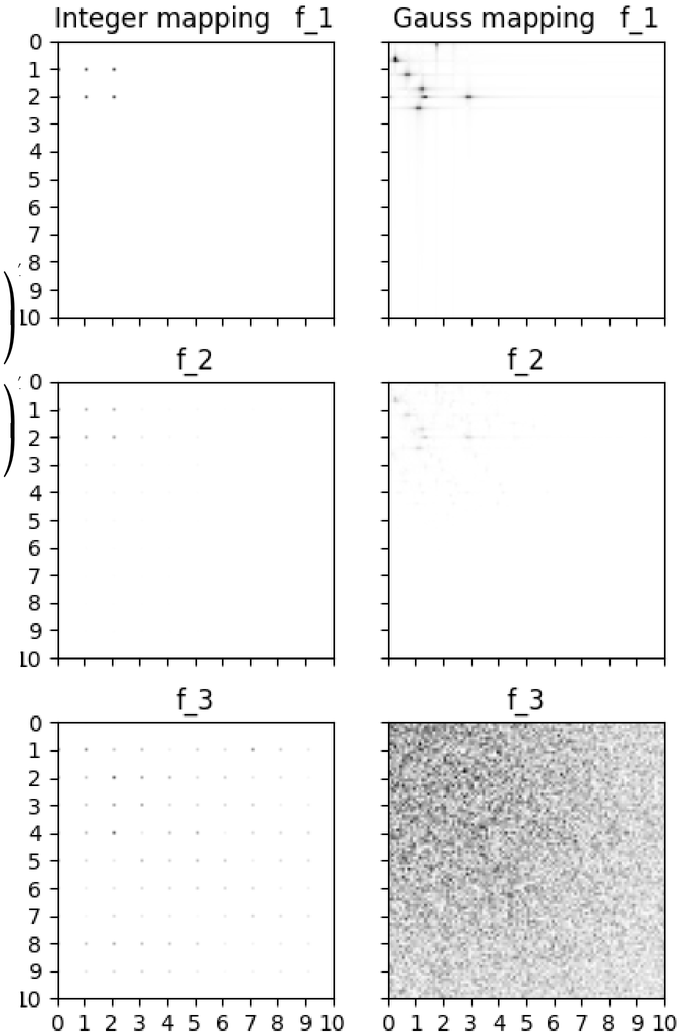

Figure 2: The effect of the common activation functions on the spectrums of functions with integer and non-integer frequencies.

We see that when a non-linearity is applied to the function with integer frequencies, the output spectrum has only integer frequencies, and this means that it is periodic. Also mentionable is the beautiful alignment of the high frequencies, which is in contrast to the Gauss mapping, where no clear pattern is present. Moreover, we see that the sine activation produces more high frequency components than ReLUs, and this could explain why sine activations are more effective at shallow networks.