Supplementary Material for the Paper: HyperPosePDF Hypernetworks Predicting the Probability Distribution on SO(3)

1. Mathematical Calculus

1.1. Derivation of equation (5)

We assume the rotation to be given in Euler coordinates. An explicit expression for the Wigner-D functions is given by:

$$D_l^{m,n}(\alpha,\beta,\gamma) = e^{-im\alpha}e^{-in\gamma}d_l^{m,n}(\cos(\beta)),$$

where

$$\begin{aligned} d_l^{m,n} =& \frac{(-1)^{l-m}}{2^l}\sqrt{\frac{(l+m)!}{(l-n)!(l+n)!(l-m)!}} \\ &\cdot\sqrt{\frac{(1-x)^{n-m}}{(1+x)^{m+n}}}\frac{\mathrm{d}^{l-m}}{\mathrm{d}\,x^{l-m}}\frac{(1+x)^{n+l}}{(1-x)^{n-l}}. \end{aligned}$$

Hence, the Fourier sum reads as

$$\begin{aligned} f(\alpha,\beta,\gamma) &= \sum_{l=1}^{L}\sum_{m,n=-l}^{l} f_{l,m,n}D_l^{m,n}(\alpha,\beta,\gamma) \\ &= \sum_{l=1}^{L}\sum_{m,n=-l}^{l} f_{l,m,n}e^{-im\alpha}e^{-in\gamma}d_l^{m,n}(\cos(\beta)) \end{aligned}$$

Rearranging the sums yields

$$= \sum_{m=-L}^{L} e^{-im\alpha}\sum_{n=-L}^{L} e^{-in\gamma}\sum_{l=\max(|m|,|n|)}^{L} f_l^{m,n}d_l^{m,n}(\cos(\beta)).$$

To get rid of the sum on the right, we follow [4] to transform a linear combination of the $d_l^{m,n}$'s into a linear combination of first kind Chebychev-polynomials which we call T_l. This results in

$$= \sum_{m,n=-L}^{L} e^{-im\alpha-in\gamma}\sum_{l=0}^{B} t_l^{m,n}T_l(\cos(\beta))(\sin(\beta)^{\mathrm{mod}(m+n,2)}.$$

We choose the coefficients $h_l^{m,n}$ such that they fulfil

$$\sum_{l=0}^{L} t_l^{m,n}T_l(\cos(\beta)) = \sum_{l=-L}^{L} h_l^{m,n}e^{-il\beta},$$

if $m+n$ is even and

$$\sin(\beta)\sum_{l=0}^{L} t_l^{m,n}T_l(\cos(\beta)) = \sum_{l=-L}^{L} h_l^{m,n}e^{-il\beta},$$

if $m+n$ is odd. Together, we receive

$$f(\alpha,\beta,\gamma) = \sum_{l,m,n=-L}^{L} h_l^{m,n}e^{-i\big((m,n,l)(R(\alpha,\beta,\gamma)\big)}.$$

1.2. Derivation of equation (6)

It is possible to prove the equation not only for dimension 3 as we need it, but for arbitrary dimension $d\in\mathbb{N}$. For the ease of writing we define $\mathbf{n}=(n_1,n_2,n_3)=(l,m,n)$ and $\mathbf{x}=(\alpha,\beta,\gamma)$. Instead of writing $h_l^{m,n}$ we write $h_{n_1,n_2,n_3}=h_{\mathbf{n}}$. Following [1] it the equation can be proved via mathematical induction:

<u>$d=1$:</u>

$$\begin{aligned} f(x) &= \sum_{n\in\mathbb{Z}^1} h_n e^{2\pi in\cdot x} \\ &= \sum_{n\in\mathbb{N}} h_n e^{2\pi in\cdot x} + \sum_{n\in\mathbb{N}} c_{-n}e^{-2\pi in\cdot x} + h_0 \\ &\overset{h_n^*=c_{-n}}{=} \sum_{n\in\mathbb{N}} \big(\mathrm{Re}(h_n)+i\mathrm{Im}(h_n)\big)\big(\cos(2\pi nx)+i\sin(2\pi nx)\big) \\ &+ \sum_{n\in\mathbb{N}} \big(\mathrm{Re}(h_n)-i\mathrm{Im}(h_n)\big)\big(\cos(2\pi nx)-i\sin(2\pi nx)\big) \\ &+ h_0 \\ &= \sum_{n\in\mathbb{N}} 2\mathrm{Re}(h_n)\cos(2\pi nx) - 2\mathrm{Im}(h_n)\sin(2\pi nx) \\ &+ h_0 \\ &= \sum_{n\in\mathbb{N}_0} a_n\cos(2\pi nx) + b_n\sin(2\pi nx), \end{aligned}$$

where

$$a_0 = h_0,\ a_n = 2\mathrm{Re}(h_n),\ b_n = -2\mathrm{Im}(h_n).$$

Assumption of the induction:
We will assume that the equation holds for $d-1$, where $d \geq 2$.

Induction step: $d-1 \to d$:
As the fourier series of any periodic and continous function is absolutely convergent, we are allowed to rearrange the sum in $(*)$ and receive

$$
\begin{aligned}
&\sum_{\mathbf{n}=(n_1,\dots,n_d)\in\mathbb{Z}^d} h_{\mathbf{n}} e^{2\pi i \mathbf{n}\cdot\mathbf{x}} \\
&\overset{(*)}{=} \sum_{n_1\in\mathbb{N}} \sum_{(n_2,\dots,n_d)\in\mathbb{Z}^{d-1}} h_{\mathbf{n}} e^{2\pi i \mathbf{n}\cdot\mathbf{x}} \\
&\quad + \sum_{n_1\in\mathbb{N}} \sum_{(n_2,\dots,n_d)\in\mathbb{Z}^{d-1}} c_{-\mathbf{n}} e^{-2\pi i \mathbf{n}\cdot\mathbf{x}} \\
&\quad + \sum_{n_1=0}^{0} \sum_{(n_2,\dots,n_d)\in\mathbb{Z}^{d-1}} h_{\mathbf{n}} e^{2\pi i \mathbf{n}\cdot\mathbf{x}} \\
&\overset{h_{\mathbf{n}}^*=c_{-\mathbf{n}}}{=} \sum_{\mathbf{n}\in\mathbb{N}\times\mathbb{Z}^{d-1}} 2\mathrm{Re}(h_{\mathbf{n}})\cos(2\pi\mathbf{n}\cdot\mathbf{x}) - 2\mathrm{Im}(h_{\mathbf{n}})\sin(2\pi\mathbf{n}\cdot\mathbf{x}) \\
&\quad + \sum_{\mathbf{n}\in\{0\}\times\mathbb{Z}^{d-1}} h_{\mathbf{n}} e^{2\pi i \mathbf{n}\cdot\mathbf{x}} \\
&\overset{\text{Ind. asm.}}{=} \sum_{\mathbf{n}\in\mathbb{N}\times\mathbb{Z}^{d-1}} 2\mathrm{Re}(h_{\mathbf{n}})\cos(2\pi\mathbf{n}\cdot\mathbf{x}) - 2\mathrm{Im}(h_{\mathbf{n}})\sin(2\pi\mathbf{n}\cdot\mathbf{x}) \\
&\quad + \sum_{\mathbf{n}\in\{0\}\times\mathbb{N}_0\times\mathbb{Z}^{d-2}} a'_{\mathbf{n}}\cos(2\pi\mathbf{n}\cdot\mathbf{x}) + b'_{\mathbf{n}}\sin(2\pi\mathbf{n}\cdot\mathbf{x}),
\end{aligned}
$$

where

$$
\begin{aligned}
a'_{\mathbf{0}} &= h_{\mathbf{0}}, \\
a'_{\mathbf{n}} &= \begin{cases} 0 & \exists j\in\{3,\dots,d\}:\, n_2=\dots=n_{j-1}=0 \wedge n_j<0 \\ 2\mathrm{Re}(h_{\mathbf{n}}) & \text{otherwise,} \end{cases} \\
b'_{\mathbf{n}} &= \begin{cases} 0 & \exists j\in\{3,\dots,d\}:\, n_2=\dots=n_{j-1}=0 \wedge n_j<0 \\ -2\mathrm{Im}(h_{\mathbf{n}}) & \text{otherwise.} \end{cases}
\end{aligned}
$$

Combining these two summands we get

$$
\sum_{\mathbf{n}\in\mathbb{N}_0\times\mathbb{Z}^{d-1}} a_{\mathbf{n}}\cos(2\pi\mathbf{n}\cdot\mathbf{x}) + b_{\mathbf{n}}\sin(2\pi\mathbf{n}\cdot\mathbf{x}),
$$

where

$$
\begin{aligned}
a_{\mathbf{0}} &= h_{\mathbf{0}}, \\
a_{\mathbf{n}} &= \begin{cases} 0 & \exists j\in\{2,\dots,d\}:\, n_1=\dots=n_{j-1}=0 \wedge n_j<0 \\ 2\mathrm{Re}(h_{\mathbf{n}}) & \text{otherwise,} \end{cases} \\
b_{\mathbf{n}} &= \begin{cases} 0 & \exists j\in\{2,\dots,d\}:\, n_1=\dots=n_{j-1}=0 \wedge n_j<0 \\ -2\mathrm{Im}(h_{\mathbf{n}}) & \text{otherwise.} \end{cases}
\end{aligned}
$$

2. Related work

In this section we want to discuss the approaches of other works, that we compared to but were not discussed in the paper.

2.1. Deng et al. [2]

In their work they introduce Deep Bingham networks, a framework to handle pose ambiguities. They introduce a multi hypotheses head to predict a family of poses to capture the nature of the solution space. From a technical perspective, they regress Bingham mixture models. Here, the Bingham distribution lies on $\mathbb{S}^{d-1}$ and is an antipodally symmetric probability distribution derived from a Gaussian with zero mean.

2.2. Gilitschenksi et al. [3]

To deal with the uncertainty of orientation they introduce a loss to capture the symmetries by characterizing uncertainty with unit quaternions based on the Bingham distribution. They name their introduced loss 'Bingham loss'. Furthermore, they demonstrate multimodal orientation prediction by using a Bingham variant of mixture density networks.

2.3. Prokudin et al. [5]

They propose a probabilistic deep learning deep learning model to predict a mixture of von Mises distributions. With that they are able to to learn a mixture model using a finite and infinite number of mixture components. Furthermore, they give an analysis on the importance of probabilistic regression.

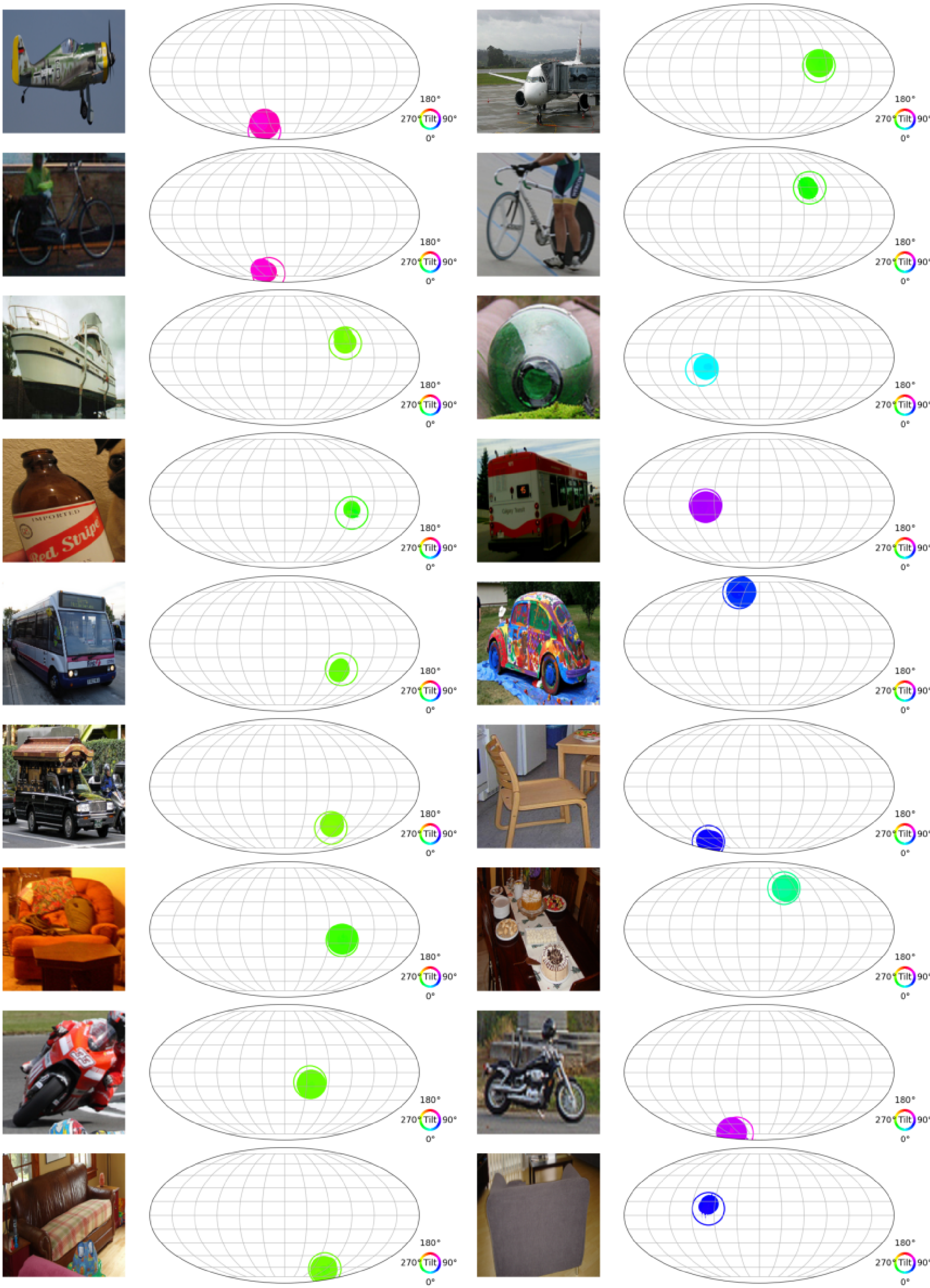
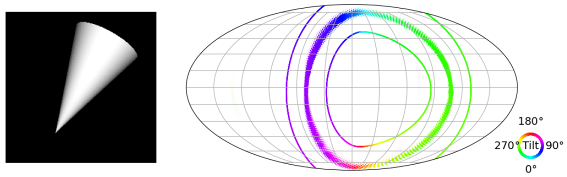
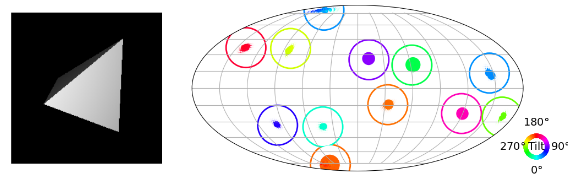


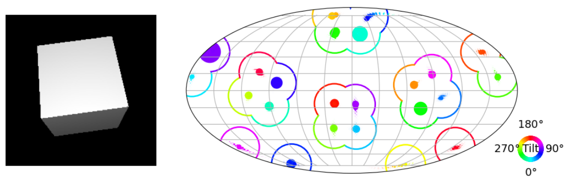
Figure 1: Further results on daily life images from the Pascal Voc dataset.



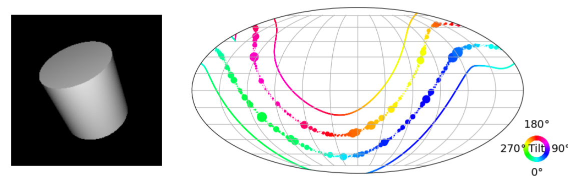
(a) No marking on the object, therefore our model predicts a continous symmetry.



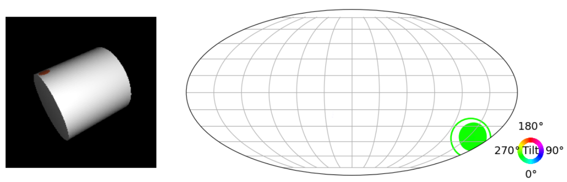
(b) Our model captures all 12 symmetries.



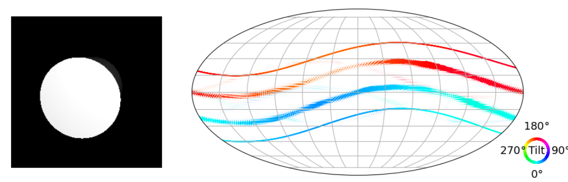
(c) The cube consists of 24 symmetries which we are able to capture.



(d) The cylinder without marker has a continous symmetry.



(e) The red marker is visible on the cylinder. Hence, no symmetries are present.



(f) The cylinder without a marker has a continuous symmetry.

Figure 2: Visualization of results on objects from the SYMSOL and SYMSOL II datasets.

References

[1] Nuri Benbarka, Timon Höfer, Andreas Zell, et al. Seeing implicit neural representations as fourier series. In *Proceedings of the IEEE/CVF Winter Conference on Applications of Computer Vision*, pages 2041–2050, 2022.

[2] Haowen Deng, Mai Bui, Nassir Navab, Leonidas Guibas, Slobodan Ilic, and Tolga Birdal. Deep bingham networks: Dealing with uncertainty and ambiguity in pose estimation. *International Journal of Computer Vision*, pages 1–28, 2022.

[3] Igor Gilitschenski, Roshni Sahoo, Wilko Schwarting, Alexander Amini, Sertac Karaman, and Daniela Rus. Deep orientation uncertainty learning based on a bingham loss. In *International Conference on Learning Representations*, 2019.

[4] Daniel Potts, Jürgen Prestin, and Antje Vollrath. A fast fourier algorithm on the rotation group. *Preprint A-07-06, Univ. zu Lübeck*, 2007.

[5] Sergey Prokudin, Peter Gehler, and Sebastian Nowozin. Deep directional statistics: Pose estimation with uncertainty quantification. In *Proceedings of the European conference on computer vision (ECCV)*, pages 534–551, 2018.