Appendix

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Abstract

This document provides additional details for the article ***Fast Diffusion EM: a diffusion model for blind inverse problems with application to deconvolution****.*

A. Iterative Diffusion EM algorithm

Algorithm A.1 summarizes the Diffusion EM algorithm described in sections 3.1 and 3.2.

Algorithm A.1 Diffusion EM algorithm

Require: $y, \sigma, H_0, L,$
Ensure: $H \approx \arg\min_H p(y|H)$ and $x_0^i \sim p(x_0|y, H)$
  for $l = 1$ **to** L **do**
    $\boldsymbol{x} = \textit{E-step}(y, H_{l-1}, \sigma)$ ▷ n samples from Alg. 1
    $H_l = \textit{M-step}(y, \boldsymbol{x}, \sigma)$ ▷ Iterate (24) and (25)
  end for
  return $\boldsymbol{x}$, H_L

B. M-step computations

In this section, we derive the computation of the M-step. In particular, we solve Equation (22) from the main paper:

$$Z^* = \arg\min_{Z \in \mathcal{C}} \frac{1}{2\sigma^2 n} \sum_{i=1}^{n} \|Zx^i - y\|_2^2 + \frac{\beta}{2}\|Z - H\|_2^2. \tag{B.1}$$

with $\mathcal{C}$ the space of convolution operators.
In order to account for the fact that $H \in \mathcal{C}$ and $Z_t \in \mathcal{C}$ are convolution operators, we rewrite the same equation in the Fourier domain, where the operators H and Z become diagonal:

$$\mathcal{F}(H) = \operatorname{diag}(h(1), \dots, h(d)), \tag{B.2}$$

$$\mathcal{F}(Z) = \operatorname{diag}(z(1), \dots, z(d)). \tag{B.3}$$

Re-writing the minimization in the Fourier domain leads to:

$$\mathcal{F}(Z^*) = \arg\min_{Z \in \mathcal{C}} \frac{1}{2\sigma^2 n} \sum_{i=1}^{n} \|\mathcal{F}(Z)\mathcal{F}(x^i) - \mathcal{F}(y)\|_2^2 + \frac{\beta}{2}\|\mathcal{F}(Z) - \mathcal{F}(H)\|_2^2 \tag{B.4}$$

$$= \arg\min_{z} \frac{1}{2\sigma^2 n} \sum_{i=1}^{n} \sum_{j=1}^{d} |z(j)\mathcal{F}(x^i)(j) - \mathcal{F}(y)(j)|^2 + \frac{\beta}{2} \sum_{j=1}^{d} |z(j) - h(j)|^2. \tag{B.5}$$

It is straightforward that the solution to the problem is also diagonal, thus we have:

$$\mathcal{F}(Z^*) = \mathrm{diag}(z^*(1), \dots, z^*(d)). \tag{B.6}$$

Using the first-order condition and the diagonal structure of the problem, we get the following:

$$\frac{1}{\sigma^2 n}\sum_{i=1}^{n}\left[z^*(j)\mathcal{F}(x^i)(j) - \mathcal{F}(y)(j)\right]\overline{\mathcal{F}(x^i)(j)} + \beta(z^*(j) - k(j)) = 0 \tag{B.7}$$

$$\Leftrightarrow z^*(j)\left(\frac{1}{n}\sum_{i=1}^{n}|\mathcal{F}(x^i)(j)|^2 + \sigma^2\beta\right) = \mathcal{F}(y)(j)\frac{1}{n}\sum_{i=1}^{n}\overline{\mathcal{F}(x^i)(j)} + \sigma^2\beta k(j) \tag{B.8}$$

$$\Leftrightarrow z^*(j) = \frac{\mathcal{F}(y)(j)\frac{1}{n}\sum_{i=1}^{n}\overline{\mathcal{F}(x^i)(j)} + \sigma^2\beta k(j)}{\frac{1}{n}\sum_{i=1}^{n}|\mathcal{F}(x^i)(j)|^2 + \sigma^2\beta}. \tag{B.9}$$

C. M-step computations with DPS approximation

In this section, we develop the computation of the M-step in Fast EM for DPS. We start from Equation (32) of the main paper:

$$\widehat{Q}(Z, Z_t) = \frac{-1}{2\sigma^2 n}\sum_{i=1}^{n}\|Z\widehat{x}_0^i(t) - y\|_2^2]. \tag{C.1}$$

Our goal is to compute:

$$Z^* = arg\min_{Z\in\mathcal{C}} -\widehat{Q}(Z, Z_t) + (\beta/2)\|Z - H\|_2^2. \tag{C.2}$$

We can notice that it is similar to Equation (B.4) with $\widehat{x}_0^i(t)$ instead of x^i. Thus we have that:

$$z^*(j) = \frac{\mathcal{F}(y)(j)\frac{1}{n}\sum_{i=1}^{n}\overline{\mathcal{F}(\widehat{x}_0^i(t))(j)} + \sigma^2\beta h(j)}{\frac{1}{n}\sum_{i=1}^{n}|\mathcal{F}(\widehat{x}_0^i(t))(j)|^2 + \sigma^2\beta}. \tag{C.3}$$

D. M-step computations with ΠGDM approximations

In this section, we develop the computation of the M-step in Fast EM for ΠGDM. We start from Equation (33) of the main paper:

$$\widehat{Q}(H, H_t) = \frac{-1}{2\sigma^2 n}\sum_{i=1}^{n}E_{x\sim\mathcal{N}(\widehat{x}_0^i(t), r_t^2)}[\|Hx - y\|_2^2]. \tag{D.1}$$

Our goal is to compute:

$$Z^* = \arg\min_{Z\in\mathcal{C}} -\widehat{Q}(Z, Z_t) + (\beta/2)\|Z - H\|_2^2. \tag{D.2}$$

Similarly to Section B, we work with diagonal operators so we have:

$$\mathcal{F}(H) = \mathrm{diag}(h(1), \dots, h(d)) \tag{D.3}$$

$$\mathcal{F}(Z) = \mathrm{diag}(z(1), \dots, z(d)). \tag{D.4}$$

and thus:

$$\mathcal{F}(Z^*) = \mathrm{diag}(z^*(1), \dots, z^*(d)). \tag{D.5}$$

We start by rewriting Equation D.1 in the Fourier domain using the fact that the Fourier transform preserves norms:

$$\mathcal{F}(Z^*) = \arg\min_{z} \frac{1}{2\sigma^2 n}\sum_{i=1}^{n}\sum_{j=1}^{d}E_{x\sim\mathcal{N}(\widehat{x}_0^i(t), r_t^2)}[|z(j)\mathcal{F}(x)(j) - \mathcal{F}(y)(j)|^2] + (\beta/2)\sum_{j=1}^{d}|z(j) - h(j)|^2. \tag{D.6}$$

We solve this problem using the first-order condition element by element since the problem is diagonal, the derivation inside the expectancy can be done using Fisher identity [1, Proposition D.4]:

$$\frac{1}{\sigma^2 n}\sum_{i=1}^{n} E_{x\sim\mathcal{N}(\widehat{x}_0^i(t),r_t^2)}[|z(j)\mathcal{F}(x)(j)-\mathcal{F}(y)(j)|\overline{\mathcal{F}(x)(j)}]+\beta(z(j)-h(j))=0 \tag{D.7}$$

$$\Leftrightarrow z(j)\left[\frac{1}{n}\sum_{i=1}^{n} E_{x\sim\mathcal{N}(\widehat{x}_0^i(t),r_t^2)}[|\mathcal{F}(x)(j)|^2]+\sigma^2\beta\right]=\mathcal{F}(y)(j)\frac{1}{n}\sum_{i=1}^{n} E_{x\sim\mathcal{N}(\widehat{x}_0^i(t),r_t^2)}[\overline{\mathcal{F}(x)(j)}]+\sigma^2\beta h(j) \tag{D.8}$$

Using the fact that the Fourier transform of a white Gaussian noise of variance σ^2 is a white Gaussian noise of variance σ^2, the expected values yield:

$$E_{x\sim\mathcal{N}(\widehat{x}_0,r_t^2)}[|\mathcal{F}(x)(j)|^2]=r_t^2+|\mathcal{F}(\widehat{x}_0)(j)|^2$$
$$E_{x\sim\mathcal{N}(\widehat{x}_0,r_t^2)}[\overline{\mathcal{F}(x)(j)}]=\overline{\mathcal{F}(\widehat{x}_0)(j)}$$

So we can conclude that:

$$z^*(j)=\frac{\mathcal{F}(y)(j)\frac{1}{n}\sum_{i=1}^{n}\overline{\mathcal{F}(\widehat{x}_0^i(t))(j)}+\sigma^2\beta h(j)}{\frac{1}{n}\sum_{i=1}^{n}|\mathcal{F}(\widehat{x}_0^i(t))(j)|^2+r_t^2+\sigma^2\beta} \tag{D.9}$$

The main difference with DPS approximation is that we have an extra term in the denominator r_t^2.

E. Additional results

See Figure E.1.

References

[1] Randal Douc, Eric Moulines, and David Stoffer. *Nonlinear Time Series*. Chapman and Hall/CRC, jan 2014. 3

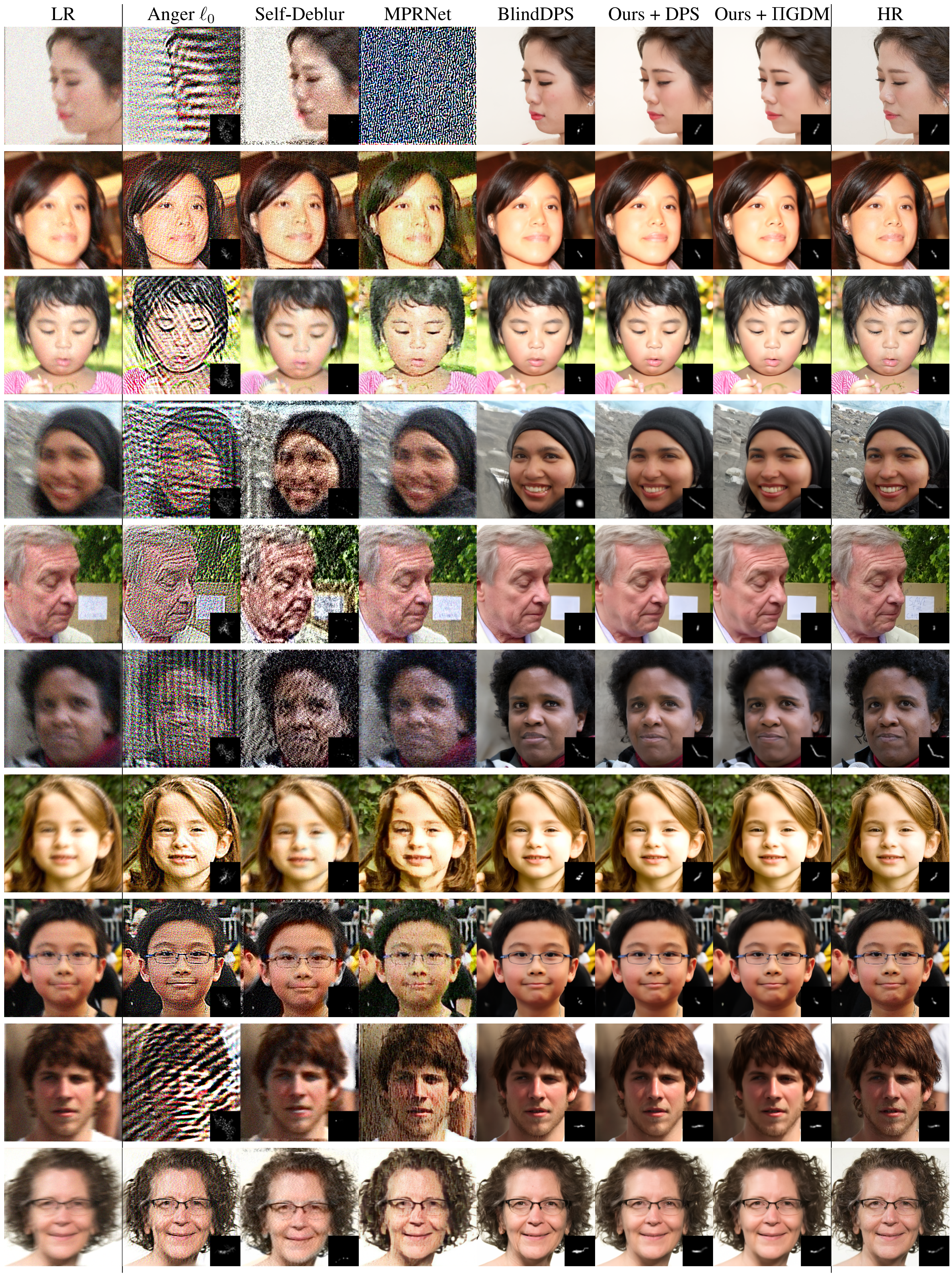

Figure E.1. Visual comparison of the different models on a degraded version of FFHQ 256x256 dataset. Ours correspond to Fast EM.