HELA-VFA: A Hellinger Distance-Attention-based Feature Aggregation Network for Few-Shot Classification

Gao Yu Lee, Tanmoy Dam, Daniel Puiu Poenar, Vu N. Duong
Nanyang Technological University (NTU), Singapore
{GAOYU001, tanmoy.dam, EPDPuiu, vu.duong}@ntu.edu.sg

Md Meftahul Ferdaus
University of New Orleans, USA
ferdaus57@gmail.com

In this supplementary material a mathematical derivation of equation (7) in the main paper, "HELA-VFA: A Hellinger Distance-Attention-based Feature Aggregation Network for Few Shot Classification" is provided, which relates the Evidence Lower Bound (ELBO) with the evidence term $\int \log(p_\theta(\mathcal{T}))dz$ and the Hellinger distance D_H.

1. Mathematical Derivation

Recall in the paper that the ELBO can be written as

$$ELBO = \int q_\phi(z|\mathcal{S})\log\left(\frac{p_\theta(\mathcal{T},z)}{q_\phi(z|\mathcal{S})}\right)dz. \tag{1}$$

and that the square of D_H is written as

$$D_H^2 = 1 - \int\left(\sqrt{p_\theta(z|\mathcal{T})q_\phi(z|\mathcal{S})}\right)dz. \tag{2}$$

Exchanging the positions of the terms on both sides and taking the logarithm of the terms,

$$\log\int\left(\sqrt{p_\theta(z|\mathcal{T})q_\phi(z|\mathcal{S})}\right)dz = \log\left(1-D_H^2\right). \tag{3}$$

Inserting the logarithm inside the integral on the left-hand side while bringing out the factor of $\frac{1}{2}$ and moving it to the term on the right-hand side,

$$\int \log\left(p_\theta(z|\mathcal{T})q_\phi(z|\mathcal{S})\right)dz = 2\log\left(1-D_H^2\right). \tag{4}$$

Noting that $\log(p_\theta(z|\mathcal{T})q_\phi(z|\mathcal{S})) = \log(p_\theta(z|\mathcal{T})) + \log(q_\phi(z|\mathcal{S}))$, the left-hand side becomes

$$\int \log(p_\theta(z|\mathcal{T}))dz + \int \log(q_\phi(z|\mathcal{S}))dz = 2\log\left(1-D_H^2\right). \tag{5}$$

Inserting the positive and negative of the evidence term on the left-hand side:

$$\begin{aligned}\int \log(p_\theta(z|\mathcal{T}))dz &+ \int \log(q_\phi(z|\mathcal{S})))dz \\ +\int \log(p_\theta(\mathcal{T}))dz &- \int \log(p_\theta(\mathcal{T}))dz \\ &= 2\log\left(1-D_H^2\right),\end{aligned} \tag{6}$$

the following is obtained,

$$\begin{aligned}\int \log(p_\theta(z,\mathcal{T}))dz &+ \int \log\left(\frac{q_\phi(z|\mathcal{S})}{p_\theta(\mathcal{T})}\right)dz \\ &= 2\log\left(1-D_H^2\right),\end{aligned} \tag{7}$$

where $p_\theta(z,\mathcal{T}) = p_\theta(z|\mathcal{T})p_\theta(\mathcal{T})$. Inserting $\frac{q_\phi(z|\mathcal{S})}{q_\phi(z|\mathcal{S})}$ in the parenthesis of $\log(p_\theta(z,\mathcal{T}))$ and expanding the logarithm,

$$\begin{aligned}\int \log\left(\frac{p_\theta(z,\mathcal{T})}{q_\phi(z|\mathcal{S})}\right)dz &+ \int \log\left(\frac{q_\phi^2(z|\mathcal{S})}{p_\theta(\mathcal{T})}\right)dz \\ &= 2\log\left(1-D_H^2\right),\end{aligned} \tag{8}$$

where the second term on the left can be expanded out to obtain

$$\begin{aligned}\int \log\left(\frac{p_\theta(z,\mathcal{T})}{q_\phi(z|\mathcal{S})}\right)dz &+ 2\int \log(q_\phi(z|\mathcal{S}))dz \\ -\int \log(p_\theta(\mathcal{T}))dz &= 2\log\left(1-D_H^2\right).\end{aligned} \tag{9}$$

Inserting the fraction $\frac{p_\theta(z,\mathcal{T})}{p_\theta(z,\mathcal{T})}$ in the parenthesis of $\log(q_\phi(z|\mathcal{S}))$, the term $2\int \log(q_\phi(z|\mathcal{S}))dz$ can be written as

$$\begin{aligned}2\int \log(q_\phi(z|\mathcal{S}))dz &= 2\int \log(p_\theta(z,\mathcal{T}))dz \\ &+2\int \log\left(\frac{q_\phi(z|\mathcal{S})}{p_\theta(z,\mathcal{T})}\right)\end{aligned} \tag{10}$$

in which the second term on the right is the inverse, and consequently, the negative of the twice of the integral of the ELBO logarithmic term, i.e., $-2\int \log\left(\frac{p_\theta(z,\mathcal{T})}{q_\phi(z|\mathcal{S})}\right)dz$. Therefore

$$\int \log\left(\frac{p_\theta(z,\mathcal{T})}{q_\phi(z|\mathcal{S})}\right)dz - 2\int \log\left(\frac{p_\theta(z,\mathcal{T})}{q_\phi(z|\mathcal{S})}\right) + 2\int \log(p_\theta(z,\mathcal{T}))dz - \int \log(p_\theta(\mathcal{T}))dz = 2\log\left(1 - D_H^2\right). \tag{11}$$

$p_\theta(z,\mathcal{T}) = p_\theta(z|\mathcal{T})p_\theta(\mathcal{T})$ is utilized again, this time on the $\log(p_\theta(z,\mathcal{T}))$ term, to obtain

$$-\int \log\left(\frac{p_\theta(z,\mathcal{T})}{q_\phi(z|\mathcal{S})}\right)dz + 2\int \log(p_\theta(z|\mathcal{T})p_\theta(\mathcal{T}))dz - \int \log(p_\theta(\mathcal{T}))dz = 2\log\left(1 - D_H^2\right), \tag{12}$$

in which the second term can be expanded out to obtain

$$-\int \log\left(\frac{p_\theta(z,\mathcal{T})}{q_\phi(z|\mathcal{S})}\right)dz + 2\int \log(p_\theta(z|\mathcal{T}))dz + 2\int \log(p_\theta(\mathcal{T}))dz - \int \log(p_\theta(\mathcal{T}))dz = 2\log\left(1 - D_H^2\right). \tag{13}$$

Now, using the fact that $p_\theta(z|\mathcal{T}) = \frac{p_\theta(z,\mathcal{T})}{p_\theta(\mathcal{T})}$, the $\log(p_\theta(z|\mathcal{T}))$ term can be written as $\log\left(\frac{p_\theta(z,\mathcal{T})}{p_\theta(\mathcal{T})}\right)$, which then along with the other terms, can be expressed as

$$-\int \log\left(\frac{p_\theta(z,\mathcal{T})}{q_\phi(z|\mathcal{S})}\right)dz + 2\int \log\left(\frac{p_\theta(z,\mathcal{T})}{p_\theta(\mathcal{T})}\right)dz + \int \log(p_\theta(\mathcal{T}))dz = 2\log\left(1 - D_H^2\right), \tag{14}$$

$$-\int \log\left(\frac{p_\theta(z,\mathcal{T})}{q_\phi(z|\mathcal{S})}\right)dz + 2\int \log\left(p_\theta(z,\mathcal{T})\right)dz - \int \log(p_\theta(\mathcal{T}))dz = 2\log\left(1 - D_H^2\right), \tag{15}$$

$$-\int \log\left(\frac{p_\theta(z,\mathcal{T})}{q_\phi(z|\mathcal{S})}\right)dz + 2\int \log\left(p_\theta(z,\mathcal{T})\right)dz = \int \log(p_\theta(\mathcal{T}))dz + 2\log\left(1 - D_H^2\right). \tag{16}$$

Finally, noting the observation that $\int \log\left(\frac{p_\theta(z,\mathcal{T})}{q_\phi(z|\mathcal{S})}\right)dz$ can be rewritten in terms of ELBO and $q_\phi(z|\mathcal{S})$ as

$$\int \log\left(\frac{p_\theta(z,\mathcal{T})}{q_\phi(z|\mathcal{S})}\right)dz = \int \left(\frac{1}{q_\phi(z|\mathcal{S})}\frac{d(ELBO)}{dz}\right)dz = \frac{1}{q_\phi(z|\mathcal{S})}ELBO, \tag{17}$$

the following is derived:

$$ELBO' = \int \log(p_\theta(\mathcal{T}))dz + \log\left(1 - D_H^2\right)^2 \tag{18}$$

where $ELBO' = \frac{ELBO}{q_\phi(z|\mathcal{S})}$.