Supplementary Material:
PULSE: Physiological Understanding with Liquid Signal Extraction

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1. Supplementary Material

We present supplementary material that enhances the understanding of our main paper through additional details and in-depth qualitative analysis. This supplementary content is structured as follows:

- Numerical Solution Using Euler's Method
- Exact Solutions of the ODE
- Gradient Derivations
- Error metric calculation

1.1. Numerical Solution Using Euler's Method:

Euler's method is a straightforward numerical technique for solving ordinary differential equations (ODEs) with a given initial value. It approximates the solution by iteratively advancing the solution over small time steps.

Problem Statement: Given the ODE:

$$\tau \frac{dh}{dt} = -h + f(x,h) + b \tag{1}$$

we want to find the numerical solution for $h(t)$ over time using Euler's method.

Steps for Solving the ODE:

1. **Rewrite the ODE**:

$$\frac{dh}{dt} = \frac{-h + f(x,h) + b}{\tau} \tag{2}$$

2. **Discretize the Time Variable**: Let's discretize the time domain into small time steps of size Δt:

$$t_n = n\Delta t \tag{3}$$

where n is an integer index representing the time step.

3. **Approximate the Derivative**: The time derivative $\frac{dh}{dt}$ at time t_n can be approximated using the forward difference:

$$\left.\frac{dh}{dt}\right|_{t=t_n} \approx \frac{h_{n+1} - h_n}{\Delta t} \tag{4}$$

4. **Substitute the Approximation into the ODE**: Substituting the forward difference approximation into the ODE, we get:

$$\frac{h_{n+1} - h_n}{\Delta t} = \frac{-h_n + f(x_n, h_n) + b}{\tau} \tag{5}$$

5. **Solve for** h_{n+1}: Rearrange the equation to solve for h_{n+1}:

$$h_{n+1} = h_n + \Delta t \cdot \frac{-h_n + f(x_n, h_n) + b}{\tau} \tag{6}$$

6. **Iterative Update Rule**: This gives us the iterative update rule for h:

$$h_{n+1} = h_n + \frac{\Delta t}{\tau}\left(-h_n + f(x_n, h_n) + b\right) \tag{7}$$

1.2. Exact Solutions of the ODE

Exact Solution Using Differential Calculus: For a simplified case, where $f(x,h)$ is a linear function of x and h, the ODE can be solved exactly. Consider:

$$\tau \frac{dh}{dt} = -h + W_{ih}x + W_{hh}h + b \tag{8}$$

Rewriting, we get:

$$\frac{dh}{dt} = -\frac{h}{\tau} + \frac{W_{ih}x + W_{hh}h + b}{\tau} \tag{9}$$

This can be solved using the integrating factor method. Multiply both sides by $e^{t/\tau}$:

$$e^{t/\tau}\frac{dh}{dt}+\frac{he^{t/\tau}}{\tau}=\frac{(W_{ih}x+W_{hh}h+b)e^{t/\tau}}{\tau} \tag{10}$$

The left side is the derivative of $he^{t/\tau}$:

$$\frac{d}{dt}\left(he^{t/\tau}\right)=\frac{(W_{ih}x+W_{hh}h+b)e^{t/\tau}}{\tau} \tag{11}$$

Integrating both sides:

$$he^{t/\tau}=\int\frac{(W_{ih}x+W_{hh}h+b)e^{t/\tau}}{\tau}\,dt \tag{12}$$

Solving for h:

$$h(t)=\left(\int\frac{(W_{ih}x+W_{hh}h+b)e^{t/\tau}}{\tau}\,dt\right)e^{-t/\tau} \tag{13}$$

1.3. Gradient Derivations

The iterative update of the hidden state h_j is defined as:

$$h^{(l)}_{t+\Delta t}=h^{(l)}_t+\frac{\Delta t}{\tau}\Bigg(-h^{(l)}_t + \tanh\left(\text{conv3d}(X,W_{ih})+\text{conv3d}(h^{(l)}_t,W_{hh})+b\right)\Bigg) \tag{14}$$

1.3.1 Gradient of Loss with respect to W_{ih}

The gradient of the loss $\mathcal{L}$ with respect to W_{ih} can be computed as follows:

$$\frac{\partial\mathcal{L}}{\partial W_{ih}}=\sum_{l=0}^{L-1}\frac{\partial\mathcal{L}}{\partial h^{(l+1)}_t}\cdot\frac{\partial h^{(l+1)}_t}{\partial W_{ih}} \tag{15}$$

The term $\frac{\partial h_{j+1}}{\partial W_{ih}}$ can be expanded using the chain rule:

$$\frac{\partial h^{(l+1)}_t}{\partial W_{ih}}=\frac{\Delta t}{\tau}\cdot\text{diag}\left(1-\tanh^2\left(\text{conv3d}(X,W_{ih})+\text{conv3d}(h^{(l)}_t,W_{hh})+b\right)\right)\cdot\frac{\partial}{\partial W_{ih}}\text{conv3d}(X,W_{ih}) \tag{16}$$

Thus,

$$\frac{\partial\mathcal{L}}{\partial W_{ih}}=\sum_{j=0}^{L-1}\frac{\partial\mathcal{L}}{\partial h^{(l+1)}_t}\cdot\frac{\Delta t}{\tau}\cdot\text{diag}\left(1-\tanh^2\left(\text{conv3d}(X,W_{ih})+\text{conv3d}(h^{(l)}_t,W_{hh})+b\right)\right)\cdot\frac{\partial}{\partial W_{ih}}\text{conv3d}(X,W_{ih}) \tag{17}$$

1.3.2 Gradient of Loss with respect to W_{hh}

Similarly, for the recurrent weights W_{hh}:

$$\frac{\partial\mathcal{L}}{\partial W_{hh}}=\sum_{l=0}^{L-1}\frac{\partial\mathcal{L}}{\partial h^{(l+1)}_t}\cdot\frac{\partial h^{(l+1)}_t}{\partial W_{hh}} \tag{18}$$

Where:

$$\frac{\partial h^{(l+1)}_t}{\partial W_{hh}}=\frac{\Delta t}{\tau}\cdot\text{diag}\left(1-\tanh^2\left(\text{conv3d}(X,W_{ih})+\text{conv3d}(h^{(l)}_t,W_{hh})+b\right)\right)\cdot\frac{\partial}{\partial W_{hh}}\text{conv3d}(h^{(l)}_t,W_{hh}) \tag{19}$$

1.3.3 Gradient of Loss with respect to τ

For τ:

$$\frac{\partial\mathcal{L}}{\partial \tau}=\sum_{l=0}^{L-1}\frac{\partial\mathcal{L}}{\partial h^{(l+1)}_t}\cdot\frac{\partial h^{(l+1)}_t}{\partial \tau} \tag{20}$$

Where:

$$\frac{\partial h^{(l+1)}_t}{\partial \tau}=-\frac{\Delta t}{\tau^2}\left(h^{(l)}_t-\tanh\left(\text{conv3d}(X,W_{ih})+\text{conv3d}(h^{(l)}_t,W_{hh})+b\right)\right) \tag{21}$$

1.3.4 Gradient of Loss with respect to b

Finally, for the bias term b:

$$\frac{\partial\mathcal{L}}{\partial b}=\sum_{l=0}^{L-1}\frac{\partial\mathcal{L}}{\partial h^{(l+1)}_t}\cdot\frac{\partial h^{(l+1)}_t}{\partial b} \tag{22}$$

Where:

$$\frac{\partial h_t^{(l+1)}}{\partial b} = \frac{\Delta t}{\tau} \cdot \text{diag}\left(1 - \tanh^2\left(\text{conv3d}(X, W_{ih}) + \text{conv3d}(h_t^{(l)}, W_{hh}) + b\right)\right) \quad (23)$$

1.4. Error metric calculation

We follow the metrics to calculate the measurement accuracy.
Here $\hat{y}$ = heart rate, y = ground truth, error $e = \hat{y} - y$

1.4.1 Pearson Correlation Coefficient (r)

The value of the Pearson correlation coefficient is between -1 to +1.

$$L_{rppg} = \frac{\frac{1}{N}\sum_{i=1}^{N}\hat{y}_i y_i - \frac{1}{N}\sum_{i=1}^{N}\hat{y}_i \frac{1}{N}\sum_{i=1}^{N} y_i}{\sqrt{\frac{1}{N}\sum_{i=1}^{N}\hat{y_i}^2 - \left(\frac{1}{N}\sum_{i=1}^{N}\hat{y}_i\right)^2}\sqrt{\frac{1}{N}\sum_{i=1}^{N}y_i^2 - \left(\frac{1}{N}\sum_{i=1}^{N}y_i\right)^2}} \quad (24)$$

1.4.2 Mean Absolute Error (MAE)

$$MAE = \sum_{i=1}^{N}\frac{|e_i|}{N} \quad (25)$$

1.4.3 Root Mean Squared Error (RMSE)

$$RMSE = \sqrt{\sum_{i=1}^{N}\frac{e_i^2}{N}} \quad (26)$$

1.4.4 Standard deviation (STD)

$$STD = \sqrt{\frac{1}{N}\sum_{i=1}^{N}e_i^2 - \left(\frac{1}{N}\sum_{i=1}^{N}e_i\right)^2} \quad (27)$$