# Supplementary Material
# A Conic Transformation Approach for Solving the Perspective-Three-Point Problem

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## 1. Detailed formulas for solving quartic equations using Ferrari's method

Consider the general quartic equation:

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0. \tag{1}$$

By dividing through by $A$ (assuming $A \neq 0$), the equation is simplified to:

$$x^4 + ax^3 + bx^2 + cx + d = 0, \tag{2}$$

where

$$a = \frac{B}{A}, b = \frac{C}{A}, c = \frac{D}{A}, d = \frac{E}{A}. \tag{3}$$

The solutions of (3) can be expressed as:

$$x = \frac{-p \pm \sqrt{p^2 - 8q}}{4}, \tag{4}$$

where $p$ and $q$ are defined as [1]:

$$\begin{aligned} q &= y_1 \mp \sqrt{y_1^2 - 4d}, \\ p &= \frac{aq \pm 2c}{\sqrt{y_1^2 - 4d}}. \end{aligned} \tag{5}$$

and $y_1$ is a real solution to the following cubic equation:

$$y^3 - by^2 + (ac - 4d)y + \left(4bd - a^2 d - c^2\right) = 0. \tag{6}$$

We check $y_1^2 - 4d$ from (5) and $p^2 - 8q$ from (4) to ensure they are greater than or equal to 0, discarding cases where they are negative. This guarantees that the solutions $x$ obtained are real numbers.

In practice, for some cases where the solutions $x$ are theoretically real, numerical computation errors may cause the value under the square root in (4) and (5) to slightly fall below zero, resulting in complex numbers. These cases are typically discarded, which may lead to the omission of some correct real solutions. As a result, when this approach is applied to a large number of cases, not all instances yield accurate solutions. For example, out of $10^7$ cases, approximately 200 may fail to provide correct results. To resolve this, we introduce an alternative method by reducing the general quartic equation in (1) to a depressed quartic form, which is a quartic equation without a cubic term. This alternative method is applied under the following two conditions: 1) The absolute value of the leading coefficient $A$ exceeds $10^4$; 2) The first method fails to yield real solutions.

We reduce the quartic equation by making the following substitution

$$x = u - \frac{B}{4A}. \tag{7}$$

Substituting this into (1), the equation transforms the equation into the following depressed quartic form [1]:

$$u^4 + au^2 + bu + c = 0, \tag{8}$$

where

$$\begin{aligned} a &= \frac{-3B^2}{8A^2} + \frac{C}{A}, \\ b &= \frac{B^3}{8A^3} - \frac{BC}{2A^2} + \frac{D}{A}, \\ c &= \frac{-3B^4}{256A^4} + \frac{CB^2}{16A^3} - \frac{BD}{4A^2} + \frac{E}{A}. \end{aligned} \tag{9}$$

The solution of (8) can be expressed as:

$$\begin{aligned} u_{1,2} &= \frac{-\xi \pm \sqrt{\xi^2 - 4(h + \zeta)}}{2}, \\ u_{3,4} &= \frac{\xi \pm \sqrt{\xi^2 - 4(h - \zeta)}}{2}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} h &= a + \eta, \\ \xi &= \sqrt{2\eta + a}, \\ \zeta &= -b/(2\sqrt{2\eta + a}), \end{aligned} \tag{11}$$

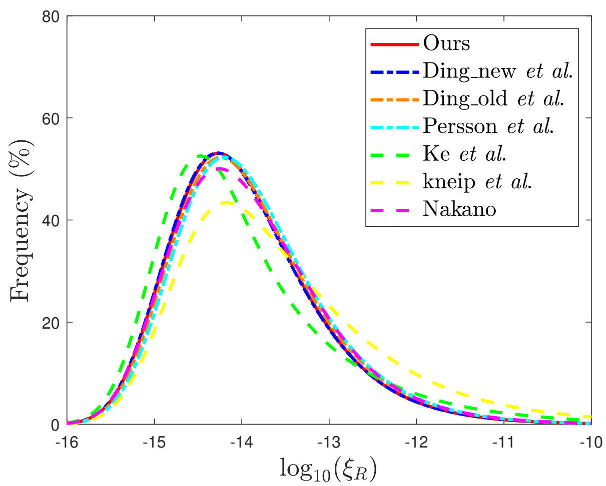
and $\eta$ is a real solution to the cubic:

$$\begin{aligned} &8y^3 + 20ay^2 + \left(16a^2 - 8c\right) y \\ &+ (4a^3 - 4ac - b^2) = 0. \end{aligned} \tag{12}$$

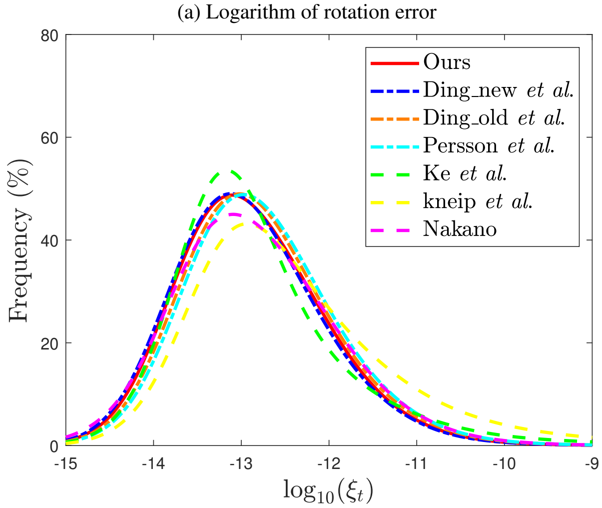
we also check $2\eta + a$ from (11), $\xi^2 - 4(h+\zeta)$, and $\xi^2 - 4(h+\zeta)$ from (10) to ensure they are greater than or equal to 0, discarding cases where they are negative.

Both methods are based on Ferrari's technique. In practice, by combining these two approaches, the quartic equation in the P3P problem can be effectively solved.

# 2. Visualizations of rotation and translation errors




(a) Logarithm of rotation error




(b) Logarithm of translation error

Figure 1. Gaussian kernel smoothed histograms of the logarithmic rotation, translation, and combined errors for various algorithms over 100,000 runs on noise-free data.

Fig. 1 shows the Gaussian kernel smoothed histograms of the logarithmic rotation error (Fig. 1a) and translation error (Fig. 1b) for various algorithms, highlighting the frequency distribution of errors across 100,000 runs on noise-free data. From Fig. 1, it can be observed that both our method and the other methods are numerically stable.

# References

[1] Girolamo Cardano, T Richard Witmer, and Oystein Ore. *The rules of algebra: Ars Magna*, volume 685. Courier Corporation, 2007. 1