Supplementary Materials for ReFu: Recursive Fusion for Exemplar-Free 3D Class-Incremental Learning

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1. Proof of Theorem 1

In phase $n-1$, we have the weight matrix as follows:

$$\hat{\boldsymbol{W}}_{n-1} = \left(\sum_{n'=0}^{n-1} \boldsymbol{A}_{n'} + \eta \boldsymbol{I}\right)^{-1} \left(\sum_{n'=0}^{n-1} \boldsymbol{C}_{n'}\right) \tag{1}$$

where $\boldsymbol{A}_{n'}$ and $\boldsymbol{C}_{n'}$ represent the auto-correlation and cross-correlation matrices. These matrices are defined as:

$$\boldsymbol{A}_{n'} = \left(\boldsymbol{F}_{n'}^{\mathrm{RP}}\right)^{\top} \boldsymbol{F}_{n'}^{\mathrm{RP}}, \quad \boldsymbol{C}_{n'} = \left(\boldsymbol{F}_{n'}^{\mathrm{RP}}\right)^{\top} \boldsymbol{Y}_{n'}^{\mathrm{train}} \tag{2}$$

Similarly, at phase n, we have:

$$\hat{\boldsymbol{W}}_{n} = \left(\sum_{n'=0}^{n} \boldsymbol{A}_{n'} + \eta \boldsymbol{I}\right)^{-1} \left(\sum_{n'=0}^{n} \boldsymbol{C}_{n'}\right) \tag{3}$$

In the paper, we define the regularized auto-correlation matrix at phase $n-1$ as:

$$\boldsymbol{R}_{n-1} = \left(\sum_{n'=0}^{n-1} \boldsymbol{A}_{n'} + \eta \boldsymbol{I}\right)^{-1} \tag{4}$$

At phase n, we have:

$$\begin{aligned} \boldsymbol{R}_{n} &= \left(\sum_{n'=0}^{n} \boldsymbol{A}_{n'} + \eta \boldsymbol{I}\right)^{-1} \\ &= \left(\sum_{n'=0}^{n-1} \boldsymbol{A}_{n'} + \boldsymbol{A}_{n} + \eta \boldsymbol{I}\right)^{-1} = \left(\boldsymbol{R}_{n-1}^{-1} + \boldsymbol{A}_{n}\right)^{-1} = \left(\boldsymbol{R}_{n-1}^{-1} + \left(\boldsymbol{F}_{n}^{\mathrm{RP}}\right)^{\top} \boldsymbol{F}_{n}^{\mathrm{RP}}\right)^{-1} \end{aligned} \tag{5}$$

According to the Woodbury matrix identity, we have:

$$(\boldsymbol{A} + \boldsymbol{U}\boldsymbol{C}\boldsymbol{V})^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{U}\left(\boldsymbol{V}\boldsymbol{A}^{-1}\boldsymbol{U} + \boldsymbol{C}^{-1}\right)\boldsymbol{V}\boldsymbol{A}^{-1} \tag{6}$$

By setting $\boldsymbol{A} = \boldsymbol{R}_{n-1}^{-1}, \boldsymbol{U} = \left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top, \boldsymbol{C} = \boldsymbol{I}$, and $\boldsymbol{V} = \boldsymbol{F}_n^{\mathrm{RP}}$ in Equation (5), this leads to the following update:

$$\boldsymbol{R}_n = \boldsymbol{R}_{n-1} - \boldsymbol{R}_{n-1}\left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top \left(\boldsymbol{F}_n^{\mathrm{RP}}\boldsymbol{R}_{n-1}\left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top + \boldsymbol{I}\right)^{-1} \boldsymbol{F}_n^{\mathrm{RP}}\boldsymbol{R}_{n-1} \tag{7}$$

This expression exemplifies the recursive nature of the updates for $\boldsymbol{R}_n$, using the prior matrix $\boldsymbol{R}_{n-1}$ and incorporating new feature data $\boldsymbol{F}_n^{\mathrm{RP}}$. This proof completes the mathematical formulation for the recursive calculation of $\boldsymbol{R}_n$.

Next, we derive the recursive calculation of $\hat{\boldsymbol{W}}_n$. To this end, we first recursively calculate $\sum_{n'=0}^{n}\boldsymbol{C}_{n'}$, i.e.,

$$\sum_{n'=0}^{n}\boldsymbol{C}_{n'} = \sum_{n'=0}^{n-1}\boldsymbol{C}_{n'} + \boldsymbol{C}_n = \sum_{n'=0}^{n-1}\boldsymbol{C}_{n'} + \left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top \boldsymbol{Y}_n^{\mathrm{train}} \tag{8}$$

Let $\boldsymbol{K}_n = \left(\boldsymbol{F}_n^{\mathrm{RP}}\boldsymbol{R}_{n-1}\left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top + \boldsymbol{I}\right)^{-1}$. Since

$$\boldsymbol{I} = \boldsymbol{K}_n\boldsymbol{K}_n^{-1} = \boldsymbol{K}_n\left(\boldsymbol{F}_n^{\mathrm{RP}}\boldsymbol{R}_{n-1}\left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top + \boldsymbol{I}\right) \tag{9}$$

we have

$$\boldsymbol{K}_n = \boldsymbol{I} - \boldsymbol{K}_n\boldsymbol{F}_n^{\mathrm{RP}}\boldsymbol{R}_{n-1}\left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top.$$

Therefore,

$$\begin{aligned}
&\boldsymbol{R}_{n-1}\left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top \left(\boldsymbol{F}_n^{\mathrm{RP}}\boldsymbol{R}_{n-1}\left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top + \boldsymbol{I}\right)^{-1} \\
&= \boldsymbol{R}_{n-1}\left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top \boldsymbol{K}_n \\
&= \boldsymbol{R}_{n-1}\left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top \left(\boldsymbol{I} - \boldsymbol{K}_n\boldsymbol{F}_n^{\mathrm{RP}}\boldsymbol{R}_{n-1}\left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top\right) \\
&= \left(\boldsymbol{R}_{n-1} - \boldsymbol{R}_{n-1}\left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top \boldsymbol{K}_n\boldsymbol{F}_n^{\mathrm{RP}}\boldsymbol{R}_{n-1}\right)\left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top \\
&= \boldsymbol{R}_n\left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top
\end{aligned} \tag{10}$$

Consequently, the updated weight matrix $\hat{\boldsymbol{W}}_n$ can be expressed as follows:

$$\begin{aligned}
\hat{\boldsymbol{W}}_n &= \boldsymbol{R}_n\sum_{n'=0}^{n}\boldsymbol{C}_{n'} \\
&= \boldsymbol{R}_n\left(\sum_{n'=0}^{n-1}\boldsymbol{C}_{n'} + \left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top \boldsymbol{Y}_n^{\mathrm{train}}\right) \\
&= \boldsymbol{R}_n\sum_{n'=0}^{n-1}\boldsymbol{C}_{n'} + \boldsymbol{R}_n\left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top \boldsymbol{Y}_n^{\mathrm{train}}
\end{aligned} \tag{11}$$

By substituting (7) into $\boldsymbol{R}_n \sum_{n'=0}^{n-1} \boldsymbol{C}_{n'}$, we have:

$$
\begin{aligned}
&\boldsymbol{R}_n \sum_{n'=0}^{n-1} \boldsymbol{C}_{n'} \\
&= \boldsymbol{R}_{n-1} \sum_{n'=0}^{n-1} \boldsymbol{C}_{n'} - \boldsymbol{R}_{n-1} \left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top \left(\boldsymbol{F}_n^{\mathrm{RP}} \boldsymbol{R}_{n-1} \left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top + \boldsymbol{I}\right)^{-1} \boldsymbol{F}_n^{\mathrm{RP}} \boldsymbol{R}_{n-1} \sum_{n'=0}^{n-1} \boldsymbol{C}_{n'} \\
&= \hat{\boldsymbol{W}}_{n-1} - \boldsymbol{R}_{n-1} \left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top \left(\boldsymbol{F}_n^{\mathrm{RP}} \boldsymbol{R}_{n-1} \left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top + \boldsymbol{I}\right)^{-1} \boldsymbol{F}_n^{\mathrm{RP}} \hat{\boldsymbol{W}}_{n-1}
\end{aligned}
\tag{12}
$$

According to (10), (12) can be rewritten as:

$$
\boldsymbol{R}_n \sum_{n'=0}^{n-1} \boldsymbol{C}_{n'} = \hat{\boldsymbol{W}}_{n-1} - \boldsymbol{R}_n \left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top \boldsymbol{F}_n^{\mathrm{RP}} \hat{\boldsymbol{W}}_{n-1} \tag{13}
$$

By inserting (13) into (11), we have:

$$
\begin{aligned}
\hat{\boldsymbol{W}}_n &= \hat{\boldsymbol{W}}_{n-1} - \boldsymbol{R}_n \left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top \boldsymbol{F}_n^{\mathrm{RP}} \hat{\boldsymbol{W}}_{n-1} + \boldsymbol{R}_n \left(\boldsymbol{F}_n^{\mathrm{RP}}\right)^\top \boldsymbol{Y}_n^{\mathrm{train}} \\
&= \hat{\boldsymbol{W}}_{n-1} - \boldsymbol{R}_n \boldsymbol{A}_n \hat{\boldsymbol{W}}_{n-1} + \boldsymbol{R}_n \boldsymbol{C}_n
\end{aligned}
\tag{14}
$$

which proves the recursive calculation of $\hat{\boldsymbol{W}}_n$.