Supplementary Material

Precise Integral in NeRFs: Overcoming the Approximation Errors of Numerical Quadrature

1. Analysis of the Accumulated Radiance

1.1. Approximation Errors of Numerical Quadrature

We provide additional clarification on the approximation error of numerical quadrature for the calculation of the accumulated radiance $C(t_i, t_{i+1})$.

Recall that $C(t_i, t_{i+1})$ is defined as follows,

$$C(t_i, t_{i+1}) = \int_{t_i}^{t_{i+1}} T(t_i, t)\sigma(\mathbf{r}(t))c(\mathbf{r}(t), \mathbf{d})\mathrm{d}t. \tag{1}$$

Let the second argument of $C(t_i, t_{i+1})$ be variable, i.e., we have function $C(t_i, t)$, then we expand it into a Taylor series at t_i as follows,

$$C(t_i, t) = C(t_i, t_i) + C^{(1)}(t_i, t_i)(t - t_i) + \frac{C^{(2)}(t_i, t_i)}{2!}(t - t_i)^2 + \cdots \tag{2}$$

$$= \sum_{n=0}^{\infty} \frac{C^{(n)}(t_i, t_i)}{n!}(t - t_i)^n. \tag{3}$$

By substituting $t = t_{i+1}$ into equation Eq. (3), we obtain a precise manner for calculating $C(t_i, t_{i+1})$ as follows,

$$\text{Precise:} \quad C(t_i, t_{i+1}) = \sum_{n=1}^{\infty} \frac{C^{(n)}(t_i, t_i)}{n!}(t_{i+1} - t_i)^n. \tag{4}$$

In the numerical quadrature, it is assumed that within the segment $[t_i, t_{i+1}]$, both $\sigma(\mathbf{r}(s))$ and $c(\mathbf{r}(s))$ are constant (i.e., they are equal to $\sigma(\mathbf{r}(t_i))$ and $c(\mathbf{r}(t_i))$ respectively). As a result, the expression for $C(t_i, t_{i+1})$ in Eq. (1) can be approximated and simplified to the form as follows,

$$\text{Quadrature:} \quad C(t_i, t_{i+1}) \approx c(\mathbf{r}(t_i), \mathbf{d}) \left(1 - e^{-\sigma(\mathbf{r}(t_i))(t_{i+1} - t_i)}\right) \tag{5}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,\sigma(\mathbf{r}(t_i))^n c(\mathbf{r}(t_i), \mathbf{d})}{n!}(t_{i+1} - t_i)^n, \tag{6}$$

where Eq. (6) is derived by utilizing $1 - e^{-x} = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!}$ in Eq. (5).

Comparing Eq. (6) to Eq. (4), we can observe that the coefficients of the Taylor series terms are different. Specifically, we calculate the first two terms of the series coefficients of Eq. (4), see below,

$$\frac{C^{(1)}(t_i, t_i)}{1!} = \frac{(-1)^2\sigma(\mathbf{r}(t_i))c(\mathbf{r}(t_i), \mathbf{d})}{1!}, \tag{7}$$

$$\frac{C^{(2)}(t_i, t_i)}{2!} = \frac{(-1)^3\sigma(\mathbf{r}(t_i))^2 c(\mathbf{r}(t_i), \mathbf{d})}{2!} \tag{8}$$

$$+ \frac{1}{2}[\sigma^{(1)}(\mathbf{r}(t_i))c(\mathbf{r}(t_i), \mathbf{d}) + \sigma(\mathbf{r}(t_i))c^{(1)}(\mathbf{r}(t_i), \mathbf{d})]. \tag{9}$$

As can be seen, the coefficients of the first term of Eq. (6) and Eq. (4) are the same (Eq. (7)), while the coefficients of the second term are different (Eq. (9)), and so are the higher terms. Therefore, the calculation method of quadrature has errors compared to the precise method.

1.2. The Unity of Theory and Practice

As mentioned in the main paper, we theoretically model $C(t_i, t_{i+1})$ as,

$$C(t_i, t_{i+1}) = \sigma_{seg}(\mathbf{r}(t_i), \mathbf{r}(t_{i+1}))c_{seg}(\mathbf{r}(t_i), \mathbf{r}(t_{i+1}))(t_{i+1} - t_i), \tag{10}$$

and in practice, we employ the following equation to calculate $C(t_i, t_{i+1})$,

$$C(t_i, t_{i+1}) = c_{seg}(\mathbf{r}(t_i), \mathbf{r}(t_{i+1}))\left(1 - e^{-\sigma_{seg}(\mathbf{r}(t_i), \mathbf{r}(t_{i+1}))(t_{i+1} - t_i)}\right) \tag{11}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sigma_{seg}(\mathbf{r}(t_i), \mathbf{r}(t_{i+1}))^n c_{seg}(\mathbf{r}(t_i), \mathbf{r}(t_{i+1})))}{n!}(t_{i+1} - t_i)^n. \tag{12}$$

As can be seen, Eq. (10) equals the first term of Eq. (12)($n = 1$); therefore Eq. (11) is a superset of our theoretical modeling. Moreover, Eq. (11) and Eq. (5) have the same overall structure, with the only difference being that each term is a unary function and a binary function, respectively. Therefore, it is very convenient to implement our method on existing approaches, most of which employ Eq. (5).

2. Per-scene Breakdown

Tabs. 1 to 6 provide a per-scene breakdown for quantity metrics in Synthetic-NeRF, NSVF, and TanksTemples datasets.

PSNR↑	chair	drums	ficus	hotdog	lego	mats.	mic	ship	mean
NeRF	34.39	24.65	30.81	36.12	34.24	30.33	34.40	29.99	31.87
PrecNeRF	35.29	25.07	31.98	36.78	36.11	31.20	35.14	30.61	32.77
NGP	35.86	25.18	33.83	37.81	36.12	29.80	37.08	31.19	33.36
PrecNGP	36.35	25.66	36.63	38.36	37.01	31.10	37.55	31.32	34.25
SSIM↑	chair	drums	ficus	hotdog	lego	mats.	mic	ship	mean
NeRF	0.977	0.924	0.970	0.979	0.974	0.956	0.987	0.885	0.957
PrecNeRF	0.982	0.929	0.975	0.981	0.981	0.963	0.989	0.888	0.961
NGP	0.986	0.932	0.982	0.983	0.981	0.950	0.992	0.900	0.963
PrecNGP	0.987	0.936	0.989	0.985	0.984	0.963	0.993	0.902	0.967
LPIPS-vgg↓	chair	drums	ficus	hotdog	lego	mats.	mic	ship	mean
NeRF	0.032	0.087	0.038	0.037	0.032	0.056	0.016	0.151	0.056
PrecNeRF	0.027	0.079	0.033	0.033	0.022	0.046	0.014	0.148	0.050
NGP	0.021	0.088	0.025	0.031	0.022	0.069	0.013	0.132	0.050
PrecNGP	0.018	0.084	0.016	0.028	0.019	0.053	0.011	0.129	0.045
LPIPS-alex↓	chair	drums	ficus	hotdog	lego	mats.	mic	ship	mean
NeRF	0.015	0.061	0.022	0.017	0.012	0.024	0.009	0.096	0.032
PrecNeRF	0.012	0.056	0.018	0.015	0.008	0.018	0.008	0.096	0.029
NGP	0.008	0.057	0.014	0.013	0.008	0.031	0.005	0.080	0.027
PrecNGP	0.007	0.053	0.008	0.011	0.007	0.020	0.005	0.077	0.023

Table 1. Quantitative results on each scene from the Synthetic-NeRF dataset

PSNR↑	Wine.	Steam.	Toad	Robot	Bike	Palace	Space.	Life.	mean
NeRF	28.89	36.20	33.21	36.91	37.92	36.98	35.27	33.40	34.85
PrecNeRF	31.43	37.98	33.43	38.69	38.81	38.06	35.92	34.16	36.06
NGP	31.87	36.37	35.89	37.43	37.90	37.56	35.69	34.72	35.93
PrecNGP	32.34	37.57	36.54	39.25	38.80	38.12	36.85	35.40	36.86
SSIM↑	Wine.	Steam	Toad	Robot	Bike	Palace	Space.	Life.	mean
NeRF	0.938	0.987	0.963	0.992	0.990	0.974	0.987	0.963	0.974
PrecNeRF	0.960	0.991	0.965	0.994	0.991	0.979	0.988	0.967	0.979
NGP	0.964	0.988	0.981	0.995	0.990	0.978	0.980	0.967	0.980
PrecNGP	0.966	0.990	0.983	0.995	0.991	0.979	0.984	0.970	0.982
LPIPS-vgg↓	Wine.	Steam	Toad	Robot	Bike	Palace	Space.	Life.	mean
NeRF	0.078	0.024	0.046	0.012	0.014	0.026	0.018	0.056	0.034
PrecNeRF	0.046	0.017	0.044	0.009	0.012	0.021	0.016	0.052	0.027
NGP	0.045	0.022	0.023	0.010	0.015	0.019	0.029	0.043	0.026
PrecNGP	0.044	0.017	0.020	0.008	0.014	0.018	0.027	0.039	0.023
LPIPS-alex↓	Wine.	Steam	Toad	Robot	Bike	Palace	Space.	Life.	mean
NeRF	0.050	0.010	0.028	0.004	0.004	0.012	0.009	0.026	0.018
PrecNeRF	0.023	0.007	0.026	0.003	0.003	0.009	0.008	0.022	0.013
NGP	0.021	0.009	0.012	0.004	0.004	0.009	0.018	0.019	0.012
PrecNGP	0.020	0.007	0.010	0.002	0.003	0.008	0.015	0.017	0.010

Table 2. Quantitative results on each scene from the NSVF dataset

PSNR↑	Ignatius	Truck	Barn	Caterpillar	Family	mean
NeRF	26.80	25.85	27.30	24.84	32.52	27.46
PrecNeRF	26.91	26.29	27.83	25.26	32.85	27.83
NGP	28.19	27.65	27.66	26.09	34.34	28.78
PrecNGP	28.48	28.23	28.53	26.38	34.51	29.23
SSIM↑	Ignatius	Truck	Barn	Caterpillar	Family	mean
NeRF	0.939	0.887	0.841	0.887	0.952	0.901
PrecNeRF	0.941	0.893	0.851	0.892	0.954	0.906
NGP	0.945	0.908	0.859	0.912	0.964	0.918
PrecNGP	0.946	0.909	0.856	0.910	0.964	0.917
LPIPS-vgg↓	Ignatius	Truck	Barn	Caterpillar	Family	mean
NeRF	0.086	0.183	0.279	0.193	0.079	0.164
PrecNeRF	0.082	0.170	0.264	0.182	0.077	0.155
NGP	0.079	0.140	0.249	0.156	0.055	0.136
PrecNGP	0.077	0.136	0.245	0.151	0.053	0.132
LPIPS-alex↓	Ignatius	Truck	Barn	Caterpillar	Family	mean
NeRF	0.082	0.153	0.225	0.160	0.060	0.136
PrecNeRF	0.076	0.137	0.206	0.146	0.055	0.124
NGP	0.071	0.111	0.209	0.115	0.038	0.109
PrecNGP	0.067	0.104	0.199	0.111	0.036	0.104

Table 3. Quantitative results on each scene from the TanksTemples dataset

	PSNR↑				SSIM↑				LPIPS-alex↓			
PyNeRF	Full	$1/2_{res}$	$1/4_{res}$	$1/8_{res}$	Full	$1/2_{res}$	$1/4_{res}$	$1/8_{res}$	Full	$1/2_{res}$	$1/4_{res}$	$1/8_{res}$
chair	35.31	38.47	39.46	39.38	0.984	0.992	0.995	0.996	0.011	0.004	0.003	0.002
drums	26.05	27.15	28.11	29.27	0.941	0.952	0.960	0.972	0.053	0.033	0.025	0.020
ficus	34.23	34.69	34.37	33.86	0.983	0.989	0.991	0.989	0.012	0.006	0.005	0.005
hotdog	37.22	39.39	40.76	41.14	0.982	0.989	0.993	0.996	0.018	0.006	0.003	0.002
lego	34.93	36.42	36.40	35.31	0.977	0.988	0.991	0.991	0.014	0.006	0.005	0.004
mats.	29.53	30.84	32.10	33.01	0.951	0.969	0.982	0.990	0.031	0.014	0.008	0.005
mic	36.38	38.59	39.70	38.97	0.991	0.995	0.995	0.997	0.006	0.003	0.003	0.002
ship	29.99	32.39	34.20	35.10	0.892	0.925	0.950	0.967	0.111	0.043	0.014	0.006
	PSNR↑				SSIM↑				LPIPS-alex↓			
PrecPyNeRF	Full	$1/2_{res}$	$1/4_{res}$	$1/8_{res}$	Full	$1/2_{res}$	$1/4_{res}$	$1/8_{res}$	Full	$1/2_{res}$	$1/4_{res}$	$1/8_{res}$
chair	35.62	38.99	40.27	40.24	0.985	0.993	0.995	0.996	0.010	0.004	0.003	0.002
drums	26.10	27.20	28.44	29.51	0.943	0.954	0.963	0.973	0.052	0.032	0.023	0.014
ficus	35.32	35.97	35.81	34.58	0.986	0.992	0.993	0.991	0.009	0.005	0.004	0.004
hotdog	37.48	39.77	41.24	41.61	0.983	0.990	0.994	0.996	0.016	0.005	0.003	0.002
lego	35.47	37.15	37.29	36.01	0.980	0.989	0.992	0.992	0.012	0.005	0.004	0.004
mats.	30.14	31.57	33.08	33.64	0.958	0.974	0.986	0.992	0.025	0.010	0.006	0.004
mic	36.54	38.78	39.98	39.40	0.992	0.995	0.995	0.997	0.006	0.003	0.002	0.002
ship	30.15	32.50	34.42	35.38	0.894	0.927	0.951	0.969	0.108	0.042	0.014	0.006

Table 4. Quantitative results on each scene from the Multiscale Synthetic-NeRF dataset

PSNR↑	chair	drums	ficus	hotdog	lego	mats.	mic	ship	mean
Constant (NGP)	35.86	25.18	33.83	37.81	36.12	29.80	37.08	31.19	33.36
Trapezoid	36.06	25.39	33.96	37.77	36.28	29.86	37.26	31.27	33.48
Parabola	35.93	25.57	34.57	37.61	36.21	29.83	37.23	31.20	33.52
PrecNGP	36.35	25.66	36.63	38.36	37.01	31.10	37.55	31.32	34.25
SSIM↑	chair	drums	ficus	hotdog	lego	mats.	mic	ship	mean
Constant (NGP)	0.986	0.932	0.982	0.983	0.981	0.950	0.992	0.900	0.963
Trapezoid	0.986	0.933	0.982	0.983	0.982	0.950	0.992	0.899	0.963
Parabola	0.986	0.935	0.984	0.983	0.981	0.949	0.992	0.899	0.964
PrecNGP	0.987	0.936	0.989	0.985	0.984	0.963	0.993	0.902	0.967
LPIPS-vgg↓	chair	drums	ficus	hotdog	lego	mats.	mic	ship	mean
Constant (NGP)	0.021	0.088	0.025	0.031	0.022	0.069	0.013	0.132	0.050
Trapezoid	0.019	0.089	0.024	0.031	0.020	0.065	0.013	0.131	0.049
Parabola	0.020	0.084	0.024	0.032	0.021	0.067	0.013	0.129	0.049
PrecNGP	0.018	0.084	0.016	0.028	0.019	0.053	0.011	0.129	0.045
LPIPS-alex↓	chair	drums	ficus	hotdog	lego	mats.	mic	ship	mean
Constant (NGP)	0.008	0.057	0.014	0.013	0.008	0.031	0.005	0.080	0.027
Trapezoid	0.008	0.059	0.013	0.013	0.007	0.030	0.005	0.080	0.027
Parabola	0.008	0.054	0.013	0.014	0.008	0.031	0.005	0.078	0.026
PrecNGP	0.007	0.053	0.008	0.011	0.007	0.020	0.005	0.077	0.023

Table 5. Quantitative results on each scene from the Synthetic-NeRF dataset when comparing PrecNGP with variants of quadrature

PSNR↑	chair	drums	ficus	hotdog	lego	mats.	mic	ship	mean
NGP(5e-3)	35.86	25.18	33.83	37.81	36.12	29.80	37.08	31.19	33.36
NGP(1e-2)	34.71	24.01	31.47	37.35	35.05	29.01	35.17	30.16	32.12
NGP(2e-2)	32.24	23.37	28.28	36.03	32.33	27.42	32.11	28.80	30.07
NGP(4e-2)	29.58	22.28	25.57	34.23	28.51	26.16	30.08	26.91	27.91
NGP(8e-2)	26.92	20.95	22.95	31.88	24.93	23.70	27.27	24.74	25.42
PrecNGP(5e-3)	36.35	25.66	36.63	38.36	37.01	31.10	37.55	31.32	34.25
PrecNGP(1e-2)	35.68	25.01	36.50	38.23	36.74	31.19	36.58	30.90	33.85
PrecNGP(2e-2)	33.95	24.59	34.42	37.57	35.05	30.51	34.30	29.86	32.53
PrecNGP(4e-2)	31.65	23.86	31.38	36.26	31.73	29.69	32.08	28.32	30.62
PrecNGP(8e-2)	29.34	22.76	26.73	34.37	27.83	28.23	29.96	26.48	28.21
SSIM↑	chair	drums	ficus	hotdog	lego	mats.	mic	ship	mean
NGP(5e-3)	0.986	0.932	0.982	0.983	0.981	0.950	0.992	0.900	0.963
NGP(1e-2)	0.981	0.916	0.974	0.981	0.977	0.944	0.988	0.891	0.956
NGP(2e-2)	0.966	0.901	0.955	0.975	0.962	0.929	0.979	0.869	0.942
NGP(4e-2)	0.940	0.878	0.929	0.965	0.923	0.910	0.966	0.839	0.919
NGP(8e-2)	0.907	0.850	0.898	0.949	0.866	0.875	0.946	0.803	0.887
PrecNGP(5e-3)	0.987	0.936	0.989	0.985	0.984	0.963	0.993	0.902	0.967
PrecNGP(1e-2)	0.985	0.932	0.988	0.984	0.983	0.964	0.991	0.898	0.966
PrecNGP(2e-2)	0.975	0.924	0.983	0.981	0.976	0.959	0.986	0.886	0.959
PrecNGP(4e-2)	0.956	0.912	0.970	0.973	0.952	0.950	0.977	0.866	0.945
PrecNGP(8e-2)	0.930	0.892	0.937	0.962	0.905	0.935	0.963	0.835	0.920
LPIPS-vgg↓	chair	drums	ficus	hotdog	lego	mats.	mic	ship	mean
NGP(5e-3)	0.021	0.088	0.025	0.031	0.022	0.069	0.013	0.132	0.050
NGP(1e-2)	0.028	0.099	0.031	0.035	0.030	0.079	0.019	0.143	0.058
NGP(2e-2)	0.049	0.124	0.051	0.048	0.059	0.089	0.033	0.174	0.078
NGP(4e-2)	0.080	0.150	0.080	0.068	0.120	0.106	0.052	0.216	0.109
NGP(8e-2)	0.112	0.190	0.110	0.102	0.178	0.136	0.075	0.263	0.146
PrecNGP(5e-3)	0.018	0.084	0.016	0.028	0.019	0.053	0.011	0.129	0.045
PrecNGP(1e-2)	0.022	0.085	0.017	0.030	0.021	0.054	0.014	0.134	0.047
PrecNGP(2e-2)	0.035	0.093	0.022	0.037	0.034	0.061	0.022	0.151	0.057
PrecNGP(4e-2)	0.059	0.112	0.037	0.052	0.077	0.073	0.037	0.180	0.078
PrecNGP(8e-2)	0.090	0.135	0.074	0.070	0.137	0.085	0.054	0.219	0.108
LPIPS-alex↓	chair	drums	ficus	hotdog	lego	mats.	mic	ship	mean
NGP(5e-3)	0.008	0.057	0.014	0.013	0.008	0.031	0.005	0.080	0.027
NGP(1e-2)	0.012	0.072	0.018	0.016	0.011	0.038	0.009	0.091	0.033
NGP(2e-2)	0.026	0.094	0.031	0.023	0.022	0.052	0.019	0.119	0.048
NGP(4e-2)	0.058	0.125	0.052	0.038	0.057	0.070	0.035	0.159	0.074
NGP(8e-2)	0.099	0.162	0.079	0.064	0.108	0.105	0.063	0.211	0.111
PrecNGP(5e-3)	0.007	0.053	0.008	0.011	0.007	0.020	0.005	0.077	0.023
PrecNGP(1e-2)	0.009	0.059	0.008	0.012	0.007	0.020	0.006	0.082	0.025
PrecNGP(2e-2)	0.016	0.066	0.012	0.016	0.012	0.025	0.011	0.097	0.032
PrecNGP(4e-2)	0.036	0.083	0.020	0.025	0.027	0.033	0.021	0.123	0.046
PrecNGP(8e-2)	0.068	0.108	0.044	0.039	0.063	0.044	0.042	0.158	0.070

Table 6. Quantitative results on each scene from the Synthetic-NeRF dataset when training NGP and PrecNGP with increasing render step size. The numbers in the parentheses below are render step size used at training time, larger render step size means fewer sampling points.