D2Mamba: Dual Domain Guided Informed Search in State Space Model for Underwater Image Enhancement - Supplementary Material

A. Theoretical Foundation: From Geodesic Distance to GIFH

This section provides the mathematical derivation connecting the classical Riemannian geodesic distance to our proposed Geodesic Information-Field Heuristic through underwater optical physics.

A.1. Riemannian Geometric Framework

We start with the basic geodesic distance formula in Riemannian geometry. The geodesic distance between two points $\mathbf{x}_1$ and $\mathbf{x}_2$ in a Riemannian manifold with metric tensor $\mathbf{g}$ is represented as:

$$d_{geo}(\mathbf{x}_1, \mathbf{x}_2) = \inf_{\gamma} \int_{\gamma} \sqrt{\dot{\boldsymbol{\gamma}}(t)^T \mathbf{g}(\boldsymbol{\gamma}(t)) \dot{\boldsymbol{\gamma}}(t)}\, dt \tag{1}$$

where $\boldsymbol{\gamma}(t)$ denotes all possible smooth curves joining the two points, and the infimum is taken over all such curves. This formulation apprehends the concept of shortest paths in curved spaces [2, 5], which serves as our basis for modeling underwater light propagation.

A.2. Connection to Fermat's Principle

The connection between Riemannian geometry and optics appears through Fermat's principle, which says that light follows paths that minimize optical path length [11]. In heterogeneous media, the optical path length is expressed as:

$$\mathcal{L}_{optical} = \int_{\gamma} n(\boldsymbol{\gamma}(t)) \|\dot{\boldsymbol{\gamma}}(t)\|\, dt \tag{2}$$

where $n(\boldsymbol{\gamma}(t))$ represents the spatially-varying refractive index. Comparing this with the geodesic distance formula, we show the fundamental connection:

$$\mathbf{g}(\boldsymbol{\gamma}(t)) = n^2(\boldsymbol{\gamma}(t))\mathbf{I} \tag{3}$$

This identification allows us to analyze the metric tensor as encoding local optical properties of the medium [6, 10], providing the connection between abstract Riemannian geometry and physical light propagation.

A.3. Physical Decomposition of Underwater Optics

Underwater environments show intricate optical behavior driven by several physical phenomena occurring at the same time. We decompose the effective refractive index into components corresponding to the three dominant underwater imaging effects. The local refractive index can be expressed as:

$$n^2(\mathbf{x}) = n_0^2 + \epsilon_G(\mathbf{x}) + \epsilon_S(\mathbf{x}) + \epsilon_A(\mathbf{x}) \tag{4}$$

where n_0 represents the base refractive index of water, and $\epsilon_G(\mathbf{x})$, $\epsilon_S(\mathbf{x})$, and $\epsilon_A(\mathbf{x})$ are spatially-varying perturbations due to curvature effects, scattering phenomena, and absorption characteristics, respectively [9, 11, 15].

A.4. First-Order Perturbation Analysis

For practical computation while maintaining physical accuracy, we use first-order perturbation theory. Considering the perturbations are small relative to the base refractive index, $|\epsilon_i| \ll n_0^2$, we can develop:

$$n(\mathbf{x}) \approx n_0 + \frac{1}{2n_0}(\epsilon_G(\mathbf{x}) + \epsilon_S(\mathbf{x}) + \epsilon_A(\mathbf{x})) \tag{5}$$

This approximation applies to typical underwater imaging scenarios, where the degradation effects, although visually significant, are relatively minor perturbations to the fundamental optical properties of the medium. [4, 16].

A.5. Discrete Path Formulation

To facilitate practical implementation within A* pathfinding algorithms, we discretize the continuous integral formulation. For paths consisting of unit-length segments, the geodesic distance becomes:

$$d_{geo}(\mathbf{x}_1, \mathbf{x}_2) \approx \sum_{i=1}^{L} n(\mathbf{x}_i) \tag{6}$$

where L represents the path length in discrete steps, and $\mathbf{x}_i$ are the discrete points along the path. This discretization retains the essential structure of the continuous formulation while allowing efficient computation on discrete pixel grids [3, 14].

A.6. Heuristic Construction for A* Search

The A* algorithm requires a heuristic function that estimates the cost from any point $\mathbf{x}$ to the goal $\mathbf{g}$. Incorporating our perturbation analysis, the heuristic becomes:

$$h(\mathbf{x}, \mathbf{g}) \approx n_0 \|\mathbf{x} - \mathbf{g}\|_2 + \frac{1}{2n_0}(\epsilon_G(\mathbf{x}) + \epsilon_S(\mathbf{x}) + \epsilon_A(\mathbf{x})) \tag{7}$$

Since the Euclidean distance term $n_0\|\mathbf{x} - \mathbf{g}\|_2$ is identical for all paths to the same goal, we can focus on the perturbation terms that distinguish different paths [12].

A.7. Physical Component Definitions

We now define each perturbation term based on established underwater imaging physics. The geodesic component captures local curvature effects:

$$\epsilon_G(\mathbf{x}) = \delta \cdot \mathcal{G}(\mathbf{x}) = \delta \cdot \frac{1}{C} \sum_{c=0}^{C-1} \|\nabla \mathbf{F}_c(\mathbf{x})\|_2 \tag{8}$$

High gradients in feature space correspond to rapid changes in medium properties, creating regions of high curvature that increase the optical path length [1]. The scattering component models information gain through texture complexity:

$$\epsilon_S(\mathbf{x}) = \gamma \cdot \mathcal{S}(\mathbf{x}, \mathbf{g}) = \gamma \cdot (\text{Var}_{HF}(\mathbf{g}) - \text{Var}_{HF}(\mathbf{x})) \tag{9}$$

Forward scattering is reduced in regions with high-frequency texture variance. Paths should prefer regions with variance characteristics similar to the goal to maintain consistent scattering properties [8, 18]. The absorption component measures low-frequency feature similarity:

$$\epsilon_A(\mathbf{x}) = \beta \cdot \mathcal{A}(\mathbf{x}, \mathbf{g}) = \beta \cdot \|\mathbf{F}_{LF}(\mathbf{x}) - \mathbf{F}_{LF}(\mathbf{g})\|_2 \tag{10}$$

Since underwater absorption primarily affects low-frequency spectral components, distance in low-frequency feature space indicates dissimilar absorption characteristics that increase optical path length [17].

A.8. Adaptive Gating Mechanism

The relative importance of scattering and absorption effects varies significantly across different underwater regions and degradation types. We introduce an adaptive gating mechanism:

$$\omega(\mathbf{x}) = \sigma(\text{MLP}(\mathbf{F}(\mathbf{x}))) \tag{11}$$

This learned function allows the model to highlight scattering costs in texture-rich regions where high-frequency preservation is crucial, while prioritizing absorption costs in smooth regions dominated by color attenuation effects [7, 13].

A.9. Final GIFH Formulation

Combining all perturbation terms with adaptive gating, and bypassing the constant Euclidean distance component, we get at our Geodesic Information-Field Heuristic:

$$h(\mathbf{x}, \mathbf{g}) = \delta \cdot \mathcal{G}(\mathbf{x}) + \omega(\mathbf{x}) \cdot \gamma \cdot \mathcal{S}(\mathbf{x}, \mathbf{g}) + (1 - \omega(\mathbf{x})) \cdot \beta \cdot \mathcal{A}(\mathbf{x}, \mathbf{g}) \tag{12}$$

This formulation maintains the fundamental principles of underwater light propagation [11, 15] and offers a computationally feasible heuristic suitable for A* pathfinding algorithms.

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