Supplementary Material – SOFEA: A Non-iterative and Robust Optical Flow Estimation Algorithm for Dynamic Vision Sensors

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1. Proof of Local Optimality of $\overline{\mathcal{N}}$

In Section 4.2 of the main text, we have claimed that the proposed greedy heuristic yields a set of neighboring events $\overline{\mathcal{N}}$ that is locally optimal with respect to the criteria laid down for $\mathcal{N}^*$. Thus, we will deliver a mathematical proof of the local optimality of $\overline{\mathcal{N}}$ in this section.

Definition 1.1. $\mathcal{N}_n^*$ is a set of neighboring events of size $n \in \mathbb{N}$ that is optimal with respect to Criterion 1 to 6. The set of all possible $\mathcal{N}_n^*$ is denoted by $\{\ \mathcal{N}_n^*\ \}$.

Definition 1.2. $\widehat{\mathcal{N}}_n$ is a set of neighboring events of size $n \in \mathbb{N}$ that is optimal with respect to Criterion 1 to 5. The set of all possible $\mathcal{N}_n^*$ is denoted by $\{\ \widehat{\mathcal{N}}_n\ \}$.

Definition 1.3. $\overline{\mathcal{N}}_n$ is the set of neighboring events of size $n \in \mathbb{N}$ that is given by the greedy heuristic.

Definition 1.4. $t_{min}(\mathcal{N})$ is a function that gives the minimum timestamp of events in the set of neighboring events $\mathcal{N}$ (*i.e.* $t_{min}(\mathcal{N}) = \min\ \{\ t \mid \boldsymbol{e} = (x, y, t, p) \in \mathcal{N}\ \}$).

Lemma 1.1 (Monotonicity Lemma)**.** *The minimum timestamp of events in $\widehat{\mathcal{N}}$ of size $n+1$ is no larger than the minimum timestamp of events in $\widehat{\mathcal{N}}$ of size n. Simply speaking, $t_{min}(\widehat{\mathcal{N}}_{n+1}) \leq t_{min}(\widehat{\mathcal{N}}_n)$.*

Proof.

1: Suppose $t_{min}(\widehat{\mathcal{N}}_{n+1}) > t_{min}(\widehat{\mathcal{N}}_n)$:
2: By removing any one of the events in $\widehat{\mathcal{N}}_{n+1}$ in such a way that Criterion 4 is not violated, we are able to obtain a new set of neighboring events $\mathcal{N}_n$ such that $t_{min}(\mathcal{N}_n) \geq t_{min}(\widehat{\mathcal{N}}_{n+1})$.
3: Combining the above with the assumption, $t_{min}(\mathcal{N}_n) > t_{min}(\widehat{\mathcal{N}}_n)$ is implied.
4: With that, there is a contradiction in the fact that $\widehat{\mathcal{N}}_n$ is optimal with respect to Criterion 1 to 5 (Definition 1.2), as $\mathcal{N}_n$ clearly provides a better solution in terms of Criterion 5.
5: Thus, $t_{min}(\widehat{\mathcal{N}}_{n+1}) \leq t_{min}(\widehat{\mathcal{N}}_n)$. ■

Definition 1.5. $\mathcal{C}^n$ is the set of candidate neighboring events $\mathcal{C}$ given by Algorithm 1 at (the end of) iteration $n \in \mathbb{N}$ of the while loop. It is equivalent to the set of 8-neighbors of all events in $\overline{\mathcal{N}}_n$. $\mathcal{C}^0$ is defined to be the initialized value of $\mathcal{C}$ in Algorithm 1 (*i.e.* $\mathcal{C}^0 = \{\ \boldsymbol{e} \mid \boldsymbol{e}$ is an 8-neighbor of $\boldsymbol{e}_{EOI}$ with the same polarity $\}$).

Definition 1.6. $\boldsymbol{e}_{new}^n = (x_{new}^n, y_{new}^n, t_{new}^n, p_{new}^n)$ is the newly selected neighboring event given by Algorithm 1 at (the end of) iteration $n \in \mathbb{N}$ of the while loop.

Theorem 1.2. *$\overline{\mathcal{N}}_n$ is optimal with respect to Criterion 1 to 5 (*i.e.* $\overline{\mathcal{N}}_n \in \{\ \widehat{\mathcal{N}}_n\ \}$)*

Proof.

1: From the greedy heuristic, it is trivial that $\overline{\mathcal{N}}_n$ satisfies Criterion 1 to 4.
2: Without loss of generality, we will make use of Algorithm 1, which provides an efficient implementation of the greedy heuristic, instead to prove that $\overline{\mathcal{N}}_n$ satisfies Criterion 5 too by Mathematical Induction.
3: **Base Case:** $k = 1$
4: Due to the constraints of Criterion 1 to 4, $\widehat{\mathcal{N}}_1 \subset \mathcal{C}^0$.
5: Along with the constraint of Criterion 5, $\widehat{\mathcal{N}}_1 = \{\ \boldsymbol{e} \mid \boldsymbol{e} \in \mathcal{C}^0$ with the largest timestamp $\}$.
6: As $\boldsymbol{e}_{new}^1$ is clearly $\boldsymbol{e} \in \mathcal{C}^0$ with the largest timestamp, we can conclude that $\overline{\mathcal{N}}_1 = \{\ \boldsymbol{e}_{new}^1\ \} \in \{\ \widehat{\mathcal{N}}_1\ \}$.
7: **Inductive Step:** $\forall k \in \mathbb{N}$
8: Assume that $\overline{\mathcal{N}}_k \in \{\ \widehat{\mathcal{N}}_k\ \}$:
9: The above assumption implies $t_{min}(\overline{\mathcal{N}}_k) = t_{min}(\widehat{\mathcal{N}}_k)$.
10: We now consider the case of $\overline{\mathcal{N}}_{k+1}$.
11: Case 1: $t_{new}^{k+1} \geq t_{min}(\overline{\mathcal{N}}_k)$
12: In this particular case, the condition of $t_{min}(\overline{\mathcal{N}}_{k+1}) = t_{min}(\overline{\mathcal{N}}_k)$, which also equals to $t_{min}(\widehat{\mathcal{N}}_k)$ (Line 9), must hold.
13: Since $t_{min}(\widehat{\mathcal{N}}_{k+1}) \leq t_{min}(\widehat{\mathcal{N}}_k)$ (Lemma 1.1: Monotonicity Lemma), the best case scenario for $\widehat{\mathcal{N}}_{k+1}$ in terms of Criterion 5 is that $t_{min}(\widehat{\mathcal{N}}_{k+1}) = t_{min}(\widehat{\mathcal{N}}_k)$.

14: This implies that $\overline{\mathcal{N}}_{k+1} \in \{ \widehat{\mathcal{N}}_{k+1} \}$, as $\overline{\mathcal{N}}_{k+1}$ achieves the best case scenario for $\widehat{\mathcal{N}}_{k+1}$ in terms of Criterion 5.

15: <u>Case 2</u>: $t_{new}^{k+1} < t_{min}(\overline{\mathcal{N}}_k)$

16: Suppose $\overline{\mathcal{N}}_{k+1} \notin \{ \widehat{\mathcal{N}}_{k+1} \}$:

17: As $\overline{\mathcal{N}}_{k+1}$ must satisfy Criterion 1 to 4, this implies that Criterion 5 is violated and hence $t_{min}(\overline{\mathcal{N}}_{k+1}) < t_{min}(\widehat{\mathcal{N}}_{k+1})$.

18: With $t_{new}^{k+1} < t_{min}(\overline{\mathcal{N}}_k)$ in this particular case, $t_{min}(\overline{\mathcal{N}}_{k+1}) = t_{new}^{k+1}$ must be true.

19: This implies that $t_{new}^{k+1} < t_{min}(\widehat{\mathcal{N}}_{k+1})$ and hence $\boldsymbol{e}_{new}^{k+1} \notin \widehat{\mathcal{N}}_{k+1}$.

20: A new set of neighboring events $\mathcal{N}_{k+1}$ can be obtained by removing a set of events $\mathcal{R}_l$ of size $l \in \{ 0, 1, \ldots, k \}$ from $\overline{\mathcal{N}}_k$ and adding in another disjoint set of events $\mathcal{A}_{l+1}$ of size $l+1$ (*i.e.* $\mathcal{N}_{k+1} = (\overline{\mathcal{N}}_k \setminus \mathcal{R}_l) \cup \mathcal{A}_{l+1}$, where $\mathcal{R}_l \subset \overline{\mathcal{N}}_k$ and $\mathcal{R}_l \cap \mathcal{A}_{l+1} = \varnothing$), in such a way that Criterion 1 to 4 are satisfied.

21: By exhaustively searching for all possible $\mathcal{N}_{k+1}$, $\widehat{\mathcal{N}}_{k+1}$ can then be found.

22: As $\mathcal{C}^k$ is the set of 8-neighbors of all k events in $\overline{\mathcal{N}}_k$ and $\mathcal{N}_{k+1}$ is of size $k+1$, at least one of the events in $\mathcal{A}_{l+1}$must be an event in $\mathcal{C}^k$.

23: Because $t_{new}^{k+1} < t_{min}(\widehat{\mathcal{N}}_k)$ (given by the condition of Case 2 and Line 9) and $\boldsymbol{e}_{new}^{k+1}$ has the largest timestamp in $\mathcal{C}^k$, $\boldsymbol{e}_{new}^{k+1}$ must be in all possible $\mathcal{N}_{k+1}$ that yields $\widehat{\mathcal{N}}_{k+1}$ (Criterion 5).

24: With that, it contradicts with the assumption that $\boldsymbol{e}_{new}^{k+1} \notin \widehat{\mathcal{N}}_{k+1}$.

25: Therefore, $\overline{\mathcal{N}}_{k+1} \in \{ \widehat{\mathcal{N}}_{k+1} \}$ is true.

26: As proven in both cases, we conclude that $\overline{\mathcal{N}}_{k+1} \in \{ \widehat{\mathcal{N}}_{k+1} \}$.

27: By Mathematical Induction, the statement is true. ■

Definition 1.7. $\widehat{\mathcal{C}}^n$ is the subset of events in $\mathcal{C}^n$ with timestamps larger than $t_{min}(\overline{\mathcal{N}}_n)$. Mathematically, $\widehat{\mathcal{C}}^n = \{ \boldsymbol{e} \mid t > t_{min}(\overline{\mathcal{N}}_n), \boldsymbol{e} = (x, y, t, p) \in \mathcal{C}^n \}$.

Corollary 1.2.1. *$\overline{\mathcal{N}}_n$ is locally optimal with respect to Criterion 1 to 6. Specifically, $\overline{\mathcal{N}}_n$ is optimal with respect to Criterion 6 (*i.e. *$\overline{\mathcal{N}}_n \in \{ \mathcal{N}_n^* \}$) in the following scenarios:*

- $\widehat{\mathcal{C}}^n = \varnothing$ *(*i.e. *$t_{min}(\mathcal{C}^n) \leq t_{min}(\overline{\mathcal{N}}_n) = t_{min}(\widehat{\mathcal{N}}_n)$), hence the sum of timestamps cannot be further maximized, while still complying with Criterion 5. This also implies a unique $\widehat{\mathcal{N}}_n$ and hence $\mathcal{N}_n^*$, given that no two events have the same timestamp.*
- $\widehat{\mathcal{C}}^n \neq \varnothing$ *(*i.e. *$t_{min}(\mathcal{C}^n) > t_{min}(\overline{\mathcal{N}}_n) = t_{min}(\widehat{\mathcal{N}}_n)$), but none of the events in $\overline{\mathcal{N}}_n$ can be removed to accommodate for events in $\widehat{\mathcal{C}}^n$, so that the sum of timestamps is further maximized, while still complying with Criterion 4.*

In other words, the two scenarios imply that $\overline{\mathcal{N}}_n$ is optimal with respect to Criterion 6 when $\mathcal{N}_n^*$ is as "spatially compact" as $\overline{\mathcal{N}}_n$, with respect to the EOI.