GPSfM: Global Projective SFM Using Algebraic Constraints on Multi-View Fundamental Matrices -Supplementary Material-

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Below we prove Theorem 1. For clarity, we base the proof on several supporting lemmas, whose proofs follow. Also, our proof of Theorem 1 relies partly on necessary conditions that were introduced and proved in [1]. Those conditions are summarized in Lemma 2.

Theorem 1. *An n-view fundamental matrix F is consistent with a set of n cameras whose centers are not all collinear if, and only if, the following conditions hold:*

1. *$Rank(F) = 6$ and F has exactly 3 positive and 3 negative eigenvalues.*
2. *$Rank(F_i) = 3$ for all $i = 1, ..., n$.*

Proof. The proof of the necessary conditions relies the properties of symmetric matrices specified in Lemma 1 and on [1], whose derivations are summarized in Lemma 2. Specifically, let F be a consistent, n-view fundamental matrix. Then, according to Lemma 2, F can be written as $F = UV^T + VU^T$, where $U, V \in \mathbb{R}^{3n\times 3}$ whose 3×3 blocks respectively are $U_i = V_iT_i$ and V_i. And, moreover, since the camera centers are not all collinear, we have $rank(F) = 6$, $rank(U) = 3$ and $rank(V) = 3$, implying property (*iii*) of Lemma 1. Consequently, using property (*i*) of Lemma 1, condition 1 holds. Condition 2 holds because not all cameras are collinear, since if conversely $rank(F_i) < 3$ for some i then there exists a 3-vector $\mathbf{e} \neq 0$ such that $F_i^T\mathbf{e} = 0$, and therefore $\forall j\ F_{ji}\mathbf{e} = 0$, i.e., all epipoles collapse to the same point in frame i, implying, in contradiction, that the camera centers are all collinear.

To establish the sufficient condition, let F be an n-view fundamental matrix that satisfies conditions 1 and 2. Condition 1 (along with property (*iii*) of Lemma 1) implies that F can be decomposed into $F = UV^T + VU^T$. This decomposition, along with condition 2, allows to deduce, using Lemma 5, that WLOG $\forall i, rank(V_i) = 3$ and $rank(U_i) = 2$. This, and the skew-symmetry of $U_iV_i^T$ (due to $F_{ii} = 0$), imply, using Lemma 4, that $V_i^{-1}U_i$ is skew-symmetric. Denote this matrix by $T_i = [\mathbf{t}_i]_\times$, we obtain $F_{ij} = V_i(T_i - T_j)V_j^T$, establishing that F is consistent. Finally, $\{\mathbf{t}_i\}_{i=1}^n$ are not all collinear, since, otherwise, by Lemma 6, $\exists i$ and $\exists \mathbf{e} \neq 0$ such that $\forall j\ F_{ji}\mathbf{e} = 0$, implying that $F_i^T\mathbf{e} = 0$, contradicting the full rank of F_i. □

We next turn to stating and proving the supporting lemmas.

Lemma 1. *Let $F \in \mathbb{S}^{3n}$ be a matrix of rank 6. Then, the following three conditions are equivalent.*

(i) *F has exactly 3 positive and 3 negative eigenvalues.*

(ii) *$F = XX^T - YY^T$ with $X, Y \in \mathbb{R}^{3n\times 3}$ and $rank(X) = rank(Y) = 3$.*

(iii) *$F = UV^T + VU^T$ with $U, V \in \mathbb{R}^{3n\times 3}$ and $rank(U) = rank(V) = 3$.*

Proof. Assume (*i*), and denote the eigenvalues of F by $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0 > \lambda_4 \geq \lambda_5 \geq \lambda_6$. Applying spectral decomposition to F we obtain

$$\begin{aligned} F &= [\tilde{X}, \tilde{Y}] \begin{pmatrix} \Sigma_1 & 0 \\ 0 & -\Sigma_2 \end{pmatrix} [\tilde{X}, \tilde{Y}]^T \\ &= \tilde{X}\Sigma_1\tilde{X}^T - \tilde{Y}\Sigma_2\tilde{Y}^T, \end{aligned}$$

where $\tilde{X}, \tilde{Y} \in \mathbb{R}^{3n\times 3}$, $\Sigma_1 = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ and $\Sigma_2 = \text{diag}(-\lambda_4, -\lambda_5, -\lambda_6)$. Next, we define $X = \tilde{X}\sqrt{\Sigma_1}$ and $Y = \tilde{Y}\sqrt{\Sigma_2}$ then

$$F = XX^T - YY^T,$$

where $rank(X) = rank(Y) = 3$, implying (*ii*). Next, let $U = \sqrt{\frac{1}{2}}(X + Y)$ and $V = \sqrt{\frac{1}{2}}(X - Y)$. It can be readily verified that

$$F = UV^T + VU^T.$$

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Moreover, if either U or V are rank deficient then $rank(F) < 6$, contradicting the assumption. Therefore, $rank(U) = rank(V) = 3$, implying (*iii*).

To complete the proof, assume (*iii*), i.e., $F = UV^T + VU^T$, where $U, V \in \mathbb{R}^{3n\times 3}$ are of rank 3. We define $X = \sqrt{\frac{1}{2}}(U+V)$ and $Y = \sqrt{\frac{1}{2}}(U-V)$ yielding $F = XX^T - YY^T$, with $rank(X) = rank(Y) = 3$, implying (*ii*).

It remains to show that (*ii*) $\Rightarrow$ (*i*). Since F is symmetric of degree 6, it has exactly 6 real, non-zero eigenvalues. We now show that exactly 3 of these eigenvalues are positive and 3 are negative. By contradiction, assume w.l.o.g. that F has at least 4 positive eigenvalues, denoted by $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, and denote their corresponding eigenvectors by v_1, v_2, v_3, v_4. Denote the subspace spanned by these eigenvectors by S, i.e., $S = span\{v_1, v_2, v_3, v_4\}$. Now, due to orthogonality, for every $\sum_{i=1}^{4} a_i v_i = z \in S$ we have

$$Fz = \sum_{i=1}^{4} \alpha_i \lambda_i v_i \Rightarrow z^T F z = \sum_{i=1}^{4} \alpha_i^2 \lambda_i.$$

Therefore, since $\lambda_i > 0$, for $0 \neq z \in S$ we have,

$$z^T F z = \sum_{i=1}^{4} \alpha_i^2 \lambda_i > 0.$$

On the other hand, the dimension of the column space of X is at most 3 and therefore $\exists \bar{z} \in S$, which is orthogonal to the column space of X, i.e. $X^T \bar{z} = 0$, implying that

$$\bar{z}^T F \bar{z} = \bar{z}^T (XX^T - YY^T)\bar{z} = -\bar{z}^T YY^T \bar{z} \leq 0,$$

which contradicts our previous observation that every vector $0 \neq z \in S$ satisfies $z^T F z > 0$. The same argument can be applied to the negative eigenvalues. We conclude that F has exactly 3 positive eigenvalues and 3 negative eigenvalues. □

Lemma 2. *[1] Let F be a consistent n-view fundamental matrix. Then,*

1. *F can be formulated as $F = UV^T + VU^T$, where $V, U \in \mathbb{R}^{3n\times 3}$ consist of n blocks of size 3×3*

$$V = \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} \quad U = \begin{bmatrix} V_1 T_1 \\ \vdots \\ V_n T_n \end{bmatrix}$$

   and $T_i = [\mathbf{t}_i]_\times$.

2. $rank(V) = 3$

3. *If $\mathbf{t}_i$ are not all collinear then $rank(U) = 3$ and $rank(F) = 6$.*

Proof. Condition 1 follows directly from Eq. (1) in the paper, namely

$$F_{ij} = V_i(T_i - T_j)V_j^T.$$

Condition 2 is satisfied since V_i is invertible for all $i = 1, ..., n$. Next, we prove Condition 3 by contradiction. Assume $rank(U) < 3$. Then, $\exists \mathbf{t} \neq 0$, s.t. $U\mathbf{t} = 0$. Since V_i are of full rank for all $i = 1, ..., n$, this implies that $\mathbf{t}_i \times \mathbf{t} = 0$ for all $i = 1, ..., n$. Thus, all the $\mathbf{t}_i$'s are parallel to $\mathbf{t}$, violating our assumption that not all $\mathbf{t}_i$ are collinear.

We are left to show that if $\mathbf{t}_i$ are not all collinear then $rank(F) = 6$. Using the QR decomposition for an invertible matrix, each V_i can be decomposed uniquely into a product of a lower triangular matrix with positive diagonal elements and an orthogonal matrix. Therefore, there exist an upper triangular K_i and an orthogonal matrix R_i such that $V_i = K_i^{-T} R_i^T$. We can thus write $F = K^T E K$, where the $3n \times 3n$ matrix K is a block diagonal matrix with 3×3 blocks formed by $\{K_i^{-1}\}_{i=1}^{n}$, and so it has full rank, implying that $rank(F) = rank(E)$. We are left to show that $rank(E) = 6$. Since E has the same form as in [1], the proof can be completed as described there ([1], p. 3). □

Lemma 3. *Let $A, B \in \mathbb{R}^{3\times 3}$such that $rank(A) = rank(B) = 2$ and $AB^T = [\mathbf{t}]_\times$ for some $\mathbf{t} \in \mathbb{R}^3$ then $A^T\mathbf{t} = B^T\mathbf{t} = 0$*

Proof. Let $\mathbf{t}_1 \in Ker(A^T)$ and $\mathbf{t}_2 \in Ker(B^T)$, $\mathbf{t}_1, \mathbf{t}_2 \neq 0$. Note also that $AB^T = [\mathbf{t}]_\times$ implies $BA^T = -[\mathbf{t}]_\times$. Then,

$$A^T\mathbf{t}_1 = 0 \Rightarrow BA^T\mathbf{t}_1 = 0 \Rightarrow -\mathbf{t} \times \mathbf{t}_1 = 0$$
$$\Rightarrow \mathbf{t}_1 \parallel \mathbf{t} \Rightarrow A^T\mathbf{t} = 0.$$

$$B^T\mathbf{t}_2 = 0 \Rightarrow AB^T\mathbf{t}_2 = 0 \Rightarrow \mathbf{t} \times \mathbf{t}_2 = 0$$
$$\Rightarrow \mathbf{t}_2 \parallel \mathbf{t} \Rightarrow B^T\mathbf{t} = 0$$

□

Lemma 4. *Let $A, B \in \mathbb{R}^{3\times 3}$ with $rank(A) = 2$, $rank(B) = 3$ and AB^T is skew symmetric (that is $AB^T + BA^T = 0$) , then $T = B^{-1}A$ is skew symmetric.*

Proof. Since AB^T is skew symmetric it can be written as $AB^T = [\mathbf{a}]_\times$ for some $\mathbf{a} \in \mathbb{R}^3 \Rightarrow$

$$\begin{aligned} A = [\mathbf{a}]_\times B^{-T} &= BB^{-1}[\mathbf{a}]_\times B^{-T} \\ &= B(B^{-1}[\mathbf{a}]_\times B^{-T}) \\ &= B\frac{[B^T\mathbf{a}]_\times}{det(B)} \end{aligned}$$

where the last equality follows from the following identity which holds for $B \in \mathbb{R}^{3\times 3}$

$$(B\mathbf{x}) \times (B\mathbf{y}) = det(B)B^{-T}(\mathbf{x} \times \mathbf{y}).$$

Consequently, $T = B^{-1}A = \frac{[B^T a]_\times}{det(B)}$ is skew symmetric. □

Lemma 5. *Let F be an n-view fundamental matrix. If F can be formulated as $F = UV^T + VU^T$ where $U, V \in \mathbb{R}^{3n\times 3}$ and in addition $rank(F_i) = 3$ for $i = 1, \dots, n$ then it holds that either $\forall i\, rank(V_i) = 3,\ rank(U_i) = 2$ or that $\forall i\, rank(V_i) = 2,\ rank(U_i) = 3$.*

Proof. First, since $\forall i\ F_{ii} = 0$, it follows that $\forall i\ U_i V_i^T$ is skew-symmetric, implying that $rank(U_i V_i^T) = 2$, and so both $2 \le rank(U_i) \le 3$ and $2 \le rank(V_i) \le 3$, but both cannot have full rank. Of the remaining possibilities.

1. $\exists i$ such that $rank(U_i) = rank(V_i) = 2$. According to Lemma 3, $\exists \mathbf{t} \in \mathbb{R}^3$, such that $U_i^T \mathbf{t} = V_i^T \mathbf{t} = 0$, implying that $F_i^T \mathbf{t} = (VU_i^T + UV_i^T)\mathbf{t} = 0$. However, this contradicts the full rank assumption over F_i.

2. Suppose, without loss of generality, that

$$rank(V_1) = 3,\ rank(U_1) = 2$$
$$rank(V_2) = 2,\ rank(U_2) = 3.$$

By Lemma 4, since $U_1 V_1^T$ is skew symmetric and $rank(V_1) = 3, rank(U_1) = 2$, we obtain that $T_1 = V_1^{-1} U_1$ is skew symmetric. By similar considerations $T_2 = U_2^{-1} V_2$ is skew symmetric. This yields

$$\begin{aligned} F_{12} &= U_1 V_2^T + V_1 U_2^T \\ &= V_1 T_1 (-T_2) U_2^T + V_1 U_2^T \\ &= V_1(-T_1 T_2 + I) U_2^T. \end{aligned}$$

Now, using the fact that $rank(V_1) = rank(U_2) = 3$, we obtain

$$rank(-T_1 T_2 + I) = rank(F_{12}) = 2. \qquad (1)$$

In the next steps we show a contradiction to (1). Since $rank(-T_1 T_2 + I) = 2$ then $\exists \mathbf{v} \in null(-T_1 T_2 + I)$, $\mathbf{v} \neq 0$ for which

$$\begin{aligned} (-T_1 T_2 + I)\mathbf{v} = 0 &\Rightarrow T_1 T_2 \mathbf{v} = \mathbf{v} \\ &\Rightarrow \mathbf{t}_1 \times (\mathbf{t}_2 \times \mathbf{v}) = \mathbf{v}. \end{aligned}$$

We conclude that $\mathbf{t}_1^T \mathbf{v} = 0$. Using the identity $a \times (b \times c) = b(a^T c) - c(a^T b)$, we obtain

$$\begin{aligned} \mathbf{t}_1 \times (\mathbf{t}_2 \times \mathbf{v}) = \mathbf{v} &\Rightarrow \\ \mathbf{t}_2(\mathbf{t}_1^T \mathbf{v}) - \mathbf{v}(\mathbf{t}_1^T \mathbf{t}_2) = \mathbf{v} &\Rightarrow \\ -\mathbf{v}(\mathbf{t}_1^T \mathbf{t}_2) = \mathbf{v} &\Rightarrow \\ (\mathbf{t}_1^T \mathbf{t}_2) = -1. \end{aligned}$$

Now, the subspace defined by $\{\mathbf{u} \in \mathbb{R}^3 | \mathbf{t}_1^T \mathbf{u} = 0\}$ is of dimension 2. However, as we show below, it is contained in $null(-T_1 T_2 + I)$, contradicting (1), since any vector $\mathbf{u}$ in this space satisfies

$$\begin{aligned} (-T_1 T_2 + I)\mathbf{u} &= -\mathbf{t}_1 \times (\mathbf{t}_2 \times \mathbf{u}) + \mathbf{u} \\ &= -\mathbf{t}_2(\mathbf{t}_1^T \mathbf{u}) + \mathbf{u}(\mathbf{t}_1^T \mathbf{t}_2) + \mathbf{u} \\ &= -\mathbf{u} + \mathbf{u} = 0. \end{aligned}$$

Consequently, either $\forall i\, rank(V_i) = 3,\ rank(U_i) = 2$ or $\forall i\, rank(V_i) = 2,\ rank(U_i) = 3$. □

For the next Lemma we note that in Lemma 5 and Theorem 1 we use the following property, which we justify below $rank(F_i) = 3 \Leftrightarrow rank(F_i^T) = 3 \Leftrightarrow null(F_i^T) = \emptyset \Leftrightarrow \nexists \mathbf{t} \in \mathbb{R}^3, \mathbf{t} \neq 0$, s.t. $F_i^T \mathbf{t} = 0 \Leftrightarrow \nexists \mathbf{t} \in \mathbb{R}^3, \mathbf{t} \neq 0$, s.t. $\forall j F_{ji} \mathbf{t} = 0$.

Lemma 6. *Let $V_1, ..., V_n \in \mathbb{R}^{3\times 3}$ and $\mathbf{t}_1, ..., \mathbf{t}_n \in \mathbb{R}^3$. We define $F_{ij} = V_i [\mathbf{t}_i - \mathbf{t}_j]_\times V_j^T$ and assume that for $i \neq j$ $rank(F_{ij}) = 2$. Then, $\{\mathbf{t}_i\}_{i=1}^n$ are collinear if and only if $\exists i \in \{1, ..., n\}$ and $\exists \mathbf{e} \in \mathbb{R}^3, \mathbf{e} \neq 0$, s.t. $\forall j F_{ji} \mathbf{e} = 0$.*

Proof. $\Rightarrow$ We first assume that $\{\mathbf{t}_i\}_{i=1}^n$ are collinear. We show it by construction. Let us choose $i \neq 1$ and define

$$\mathbf{e} = V_i^{-1}(\mathbf{t}_i - \mathbf{t}_1).$$

Since $\mathbf{t}_i \neq \mathbf{t}_1$ (otherwise the rank assumption is violated) then $\mathbf{e} \neq 0$ and the collinear points $\mathbf{t}_1, \dots, \mathbf{t}_n$ can be parameterized as follows

$$\mathbf{t}_k = \mathbf{t}_1 + \alpha_k(\mathbf{t}_i - \mathbf{t}_1)\ \ \forall k.$$

Now, $\forall j$ it holds that

$$\begin{aligned} F_{ji}\mathbf{e} &= V_j(\mathbf{t}_j - \mathbf{t}_i)_\times V_i^T V_i^{-T}(\mathbf{t}_i - \mathbf{t}_1) \\ &= V_j(\mathbf{t}_j - \mathbf{t}_i) \times (\mathbf{t}_i - \mathbf{t}_1) \\ &= V_j((\alpha_j - \alpha_i)(\mathbf{t}_i - \mathbf{t}_1)) \times (\mathbf{t}_i - \mathbf{t}_1)) = 0. \end{aligned}$$

$\Leftarrow$ Without loss of generality, we assume that $i \neq 1$. Therefore, $\exists \mathbf{e} \in \mathbb{R}^3, \mathbf{e} \neq 0$ s.t $\forall j\ F_{ji}\mathbf{e} = 0$. Since $F_{1i} = V_1[\mathbf{t}_1 - \mathbf{t}_i]_\times V_i^T \Rightarrow V_i^{-T}(\mathbf{t}_i - \mathbf{t}_1) \in null(F_{1i})$. Assuming that $rank(F_{1i}) = 2$ then the dimension of $null(F_{1i})$ is 1, implying that $\mathbf{e} = \alpha V_i^{-T}(\mathbf{t}_i - \mathbf{t}_1)$, where $\alpha \neq 0$ is a scalar. Now, $\forall j$

$$\begin{aligned} F_{ji}\mathbf{e} = 0 &\Rightarrow V_j[\mathbf{t}_j - \mathbf{t}_i]_\times V_i^T V_i^{-T}(\mathbf{t}_i - \mathbf{t}_1) = 0 \\ &\Rightarrow V_j[\mathbf{t}_j - \mathbf{t}_i]_\times (\mathbf{t}_i - \mathbf{t}_1) = 0 \\ &\Rightarrow (\mathbf{t}_j - \mathbf{t}_i) \times (\mathbf{t}_i - \mathbf{t}_1) = 0 \\ &\Rightarrow \exists \alpha_j \in \mathbb{R}\ \text{ s.t. }\ \mathbf{t}_j - \mathbf{t}_i = \alpha_j(\mathbf{t}_i - \mathbf{t}_1) \\ &\Rightarrow \mathbf{t}_j = \mathbf{t}_i + \alpha_j(\mathbf{t}_i - \mathbf{t}_1) \end{aligned}$$

concluding that the points are collinear. □

References

[1] S. Sengupta, T. Amir, M. Galun, T. Goldstein, D. W. Jacobs, A. Singer, and R. Basri. A new rank constraint on multi-view fundamental matrices, and its application to camera location recovery. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pages 4798–4806, 2017.