RES-PCA: A Scalable Approach to Recovering Low-rank Matrices

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1. Proof of Theorem 3.1

Proof. The objective function of the optimization problem in (6) is a quadratic function of Q, which can be simplified as follows:

$$\begin{aligned}&\mathbf{Tr}(Q(I-\frac{1}{n}\boldsymbol{1}\boldsymbol{1}^T)Q^T)\\&=\mathbf{Tr}(Q(I-\frac{1}{n}\boldsymbol{1}\boldsymbol{1}^T)(I-\frac{1}{n}\boldsymbol{1}\boldsymbol{1}^T)^TQ^T)\\&=\|Q(I-\frac{1}{n}\boldsymbol{1}\boldsymbol{1}^T)\|_2^2\\&=\sum_{i=1}^n\|\boldsymbol{q}_i-\bar{\boldsymbol{q}}\|_2^2,\end{aligned}$$

where $\bar{\boldsymbol{q}} := \frac{1}{n}\sum_{i=1}^n \boldsymbol{q}_i$. In the above, the second equality holds because

$$I-\frac{1}{n}\boldsymbol{1}\boldsymbol{1}^T=(I-\frac{1}{n}\boldsymbol{1}\boldsymbol{1}^T)(I-\frac{1}{n}\boldsymbol{1}\boldsymbol{1}^T)^T.$$

Now the original optimization problem can be equivalently written in vector form of $\boldsymbol{q}_i$ as follows:

$$\min_{\boldsymbol{q}_i\in\mathcal{R}^p}\sum_{i=1}^n\|\boldsymbol{q}_i-\bar{\boldsymbol{q}}\|_2^2,$$

subject to

$$\begin{aligned}&\|\boldsymbol{q}_i\|_2^2-s_i\le 0, i=1,\cdots,n,\\&\sum_{i=1}^n(\boldsymbol{q}_i-\bar{\boldsymbol{q}})=0.\end{aligned}$$

We define the Lagrangian function for the optimization function by using the new vector form in $\boldsymbol{q}_i$:

$$\mathcal{L}:=\sum_{i=1}^n\|\boldsymbol{q}_i-\bar{\boldsymbol{q}}\|_2^2+\sum_{i=1}^n\alpha_i(\boldsymbol{q}_i^T\boldsymbol{q}_i-s_i)+\boldsymbol{\beta}^T\sum_{i=1}^n(\boldsymbol{q}_i-\bar{\boldsymbol{q}}),$$

where $\alpha_i\ge 0, i=1,\cdots,n$ are dual variables for inequalities in the original contraints (7), and $\boldsymbol{\beta}\in\mathcal{R}^p$ is the dual vector for equality contraint given by the definition of $\bar{\boldsymbol{q}}$.

Taking partial derivative of $\mathcal{L}$ with respect to $\boldsymbol{q}_i$ and letting it equal to 0, we have

$$\frac{\partial\mathcal{L}}{\partial\boldsymbol{q}_i}=2(\boldsymbol{q}_i-\bar{\boldsymbol{q}})+2\alpha_i\boldsymbol{q}_i+\boldsymbol{\beta}=0,\ \text{for } i=1,\cdots,n.$$

Then we can obtain the root of the above equation:

$$\boldsymbol{q}_i^*=\frac{1}{1+\alpha_i}(\bar{\boldsymbol{q}}-\frac{1}{2}\boldsymbol{\beta}),\ \text{for } i=1,\cdots,n.$$

As the optimization problem has a quadratic objective function and quadratic constraints, Kuhn-Tucker condiction is both necessary and sufficient to achieve the optimality. Therefore, $\boldsymbol{q}_i^*, i=1,\cdots,n$, is an optimizor of the given optimization problem. Since the minimizor of the optimization problem is linearly dependent, the conclusion follows. □