Locating Objects Without Bounding Boxes - Suplemental Material

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Annex: Ablation of terms in the Weighted Hausdorff Distance

In Section 4, we made the following claim:

Claim. *Both terms of the Weigthed Hausdorff Distance (WHD) are necessary. If the first term is removed, then* $p_x = 1 \ \forall x \in \Omega$ *is the solution that minimizes the WHD. If the second term is removed, then the trivial solution is* $p_x = 0 \ \forall x \in \Omega$.

Proof. If the first term is removed and $p_x = 1 \ \forall x \in \ \Omega$, then Equation (5) reduces to

$$d_{\text{WH}}(p, Y)\big|_{p=1} = \frac{1}{|Y|} \sum_{y \in Y} \underset{x \in \Omega}{M_\alpha} \left[\, d(x, y) \,\right].$$

From the definition in Equation (2), $\forall x, y \in \Omega$,

$$d(x, y) \leq d_{max}.$$

For any $p_x \in [0, 1]$ and $\alpha < 0$,

$$\begin{aligned}
(1 - p_x) d(x, y) &\leq (1 - p_x) d_{max} \\
d(x, y) &\leq p_x d_{max} + (1 - p_x) d_{max} \\
d(x, y)^\alpha &\geq \left[p_x d_{max} + (1 - p_x) d_{max}\right]^\alpha \\
\frac{1}{|\Omega|} \sum_{x \in \Omega} d(x, y)^\alpha &\geq \frac{1}{|\Omega|} \sum_{x \in \Omega} \left[p_x d_{max} + (1 - p_x) d_{max}\right]^\alpha \\
\left[\frac{1}{|\Omega|} \sum_{x \in \Omega} d(x, y)^\alpha\right]^{\frac{1}{\alpha}} &\leq \left[\frac{1}{|\Omega|} \sum_{x \in \Omega} \left[p_x d_{max} + (1 - p_x) d_{max}\right]^\alpha\right]^{\frac{1}{\alpha}} \\
\underset{x \in \Omega}{M_\alpha} \left[\, d(x, y) \,\right] &\leq \underset{x \in \Omega}{M_\alpha} \left[\, p_x d_{max} + (1 - p_x) d_{max} \,\right] \\
\frac{1}{|Y|} \sum_{y \in Y} \underset{x \in \Omega}{M_\alpha} \left[\, d(x, y) \,\right] &\leq \frac{1}{Y} \sum_{y \in Y} \underset{x \in \Omega}{M_\alpha} \left[\, p_x d_{max} + (1 - p_x) d_{max} \,\right] \\
d_{\text{WH}}(p, Y)\big|_{p=1} &\leq d_{\text{WH}}(p, Y).
\end{aligned}$$

Note that $d_{\text{WH}}(p, Y)\big|_{p=1} > 0$ if $\alpha > -\infty$, but the proof holds for any $\alpha < 0$.
If the second term is removed and $p_x = 0 \ \forall x \in \Omega$, then Equation (5) reduces to

$$d_{\text{WH}}(p, Y)\big|_{p=0} = \frac{1}{\mathcal{S} + \epsilon} \sum_{x \in \Omega} p_x \min_{y \in Y} d(x, y)\big|_{p=0} = \frac{1}{0 + \epsilon} 0 = 0.$$

□