Fractional Anisotropy (FA)

1 Definition of FA

FA is used in neuroscience to evaluate anisotropic diffusion of water molecules in the body, and its definition is based on the diffusion equation as

$$f(\boldsymbol{g}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{D}|^{1/2}} \exp\left(-\frac{1}{2}\boldsymbol{g}^\top \boldsymbol{D}^{-1}\boldsymbol{g}\right), \tag{1}$$

where $f(\boldsymbol{g})$ is a probability distribution of water molecules in directions $\boldsymbol{g} \in \mathbb{R}^d$, and $\boldsymbol{D}$ is a positive semi-definite matrix that represents diffusion strength of the distribution $f(\boldsymbol{g})$ along or between directions $\boldsymbol{g}$. To the best of our knowledge, on the basis of $f(\boldsymbol{g})$, FA is defined in 3D case but we generalize it for multi-dimensional case as

$$FA := \sqrt{\frac{d}{d-1}} \cdot \frac{\sqrt{\sum_{i=1}^{d}(\lambda_i - \bar{\lambda})^2}}{\sqrt{\sum_{i=1}^{d}\lambda_i^2}}, \tag{2}$$

where $(\lambda_1, ..., \lambda_d)$ are eigenvalues of $\boldsymbol{D}$ and $\bar{\lambda} = \frac{1}{d}\sum_{i=1}^{d}\lambda_i$. The eigenvalues of $\boldsymbol{D}$ indicate diffusion strength to the direction of eigenvectors in original directions $\boldsymbol{g}$. $\sqrt{\frac{d}{d-1}}$ normalizes the FA value between 0 and 1.

For intuitive interpretation of FA, we attempt to rewrite the definition of FA (2) in the next section.

2 Intuitive Interpretation of FA

Lemma 1.

$$\sum_{i=1}^{d}\left(\lambda_i - \bar{\lambda}\right)^2 = \sum_{i=1}^{d}\lambda_i^2 - d \cdot \bar{\lambda}^2, \tag{3}$$

where $\bar{\lambda} = \frac{1}{d}\sum_{i=1}^{d}\lambda_i$.

Proof.

$$\begin{aligned}
\sum_{i=1}^{d}\left(\lambda_i-\bar{\lambda}\right)^2 &= \sum_{i=1}^{d}\lambda_i^2 - 2\cdot\left(\sum_{i=1}^{d}\lambda_i\right)\cdot\bar{\lambda}+d\cdot\bar{\lambda}^2\\
&= \sum_{i=1}^{d}\lambda_i^2 - 2\cdot d\cdot\bar{\lambda}^2+d\cdot\bar{\lambda}^2\\
&= \sum_{i=1}^{d}\lambda_i^2 - d\cdot\bar{\lambda}^2
\end{aligned}$$

□

Lemma 2. Let θ be the angle between two vectors, $(\lambda_1,\ldots,\lambda_d),(1,1,\ldots,1)\in\mathbb{R}^d$. Then,

$$\sin\theta=\sqrt{\frac{\sum_{i=1}^{d}\left(\lambda_i-\bar{\lambda}\right)^2}{\sum_{i=1}^{d}\lambda_i^2}}. \tag{4}$$

Proof. From the relationship between inner product and norms of two vectors, we get

$$\cos\theta=\frac{\sum_{i=1}^{d}\lambda_i}{\sqrt{\sum_{i=1}^{d}\lambda_i^2}\cdot\sqrt{\sum_{i=1}^{d}1}}=\frac{\sum_{i=1}^{d}\lambda_i}{\sqrt{\sum_{i=1}^{d}\lambda_i^2}\cdot\sqrt{d}}.$$

Because D is positive semi-definite and so the vector $(\lambda_1,\ldots,\lambda_d)$ is in the first quadrant, $\sin\theta>0$ holds. As a result, we can rewrite $\sin\theta$ as follows:

$$\begin{aligned}
\sin\theta &= \sqrt{1-\cos^2\theta}\\
&= \sqrt{1-\frac{\left(\sum_{i=1}^{d}\lambda_i\right)^2}{\left(\sum_{i=1}^{d}\lambda_i^2\right)\cdot d}}=\sqrt{1-\frac{\bar{\lambda}^2\cdot d^2}{\left(\sum_{i=1}^{d}\lambda_i^2\right)\cdot d}}=\sqrt{\frac{\sum_{i=1}^{d}\lambda_i^2-\bar{\lambda}^2\cdot d}{\sum_{i=1}^{d}\lambda_i^2}}\\
&= \sqrt{\frac{\sum_{i=1}^{d}\left(\lambda_i-\bar{\lambda}\right)^2}{\sum_{i=1}^{d}\lambda_i^2}}\quad(\because(3)).
\end{aligned}$$

□

Proposition 1.

$$FA=\sqrt{\frac{d}{d-1}}\cdot\sin\theta,$$

where θ is defined in Lemma 2.

Proof. Trivial from (2) and (4). □

Proposition 1 implies that FA purely evaluates the degree of match between the eigenvalues without depending on the magnitude of them. Since the positive semi-definite matrix of $\boldsymbol{D}$ makes all the eigenvalues positive, if only one eigenvalue is high, which means anisotropic diffusion, θ is maximum and FA value is 1, but if all the eigenvalues are equal, which means isotropic diffusion, θ is 0 and FA value is 0 (Fig.1).

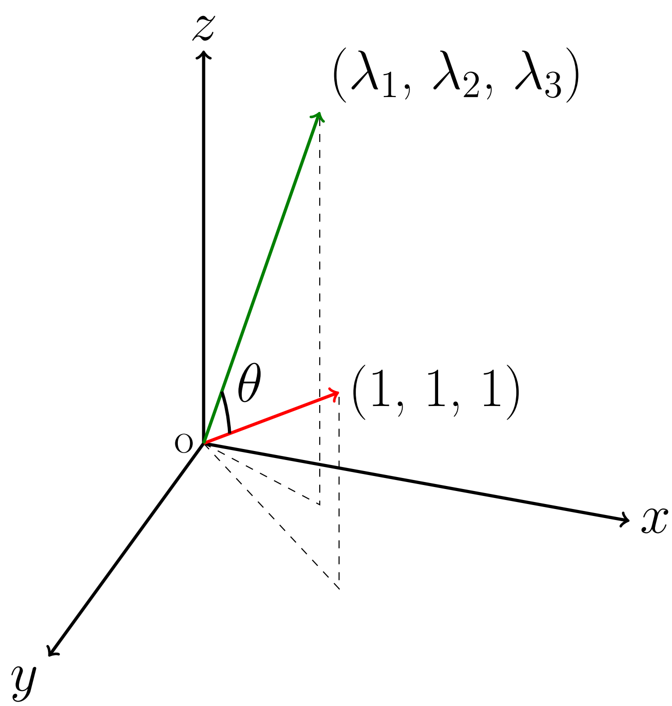

Figure 1: Intuitive interpretation of FA in 3D case. FA is proportional to sinθ, where θ is the angle between two vectors, $(\lambda_1, \lambda_2, \lambda_3)$ which are eigenvalues of $\boldsymbol{D}$, $(1, 1, 1)$. If all the eigenvalues are equal such as $(3, 3, 3)$, which means isotropic diffusion, θ is 0 and the FA value is 0. If only one eigenvalue is high such as $(5, 0, 0)$, which means anisotropic diffusion, θ is maximum and the FA value is 1.