Coordinate-Free Carlsson-Weinshall Duality and Relative Multi-View Geometry

Supplementary Material

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We present some proofs that were not included in the main part of the paper and some additional results of our algorithm on synthetic data.

1. Analytical cameras and duality

The analytical expression for $P_{\boldsymbol{c}}(\boldsymbol{x})$ given in Proposition 2.4 follows easily from the following general result.

Proposition 1.1. *The unique projective transformation of $\mathbb{P}^n$ that maps $n+2$ general points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n+2}$ to the standard basis of $\mathbb{P}^n$ can be described by*

$$\boldsymbol{y} \mapsto \begin{bmatrix} \frac{|\boldsymbol{x}_2 \ldots \boldsymbol{x}_{n+1} \boldsymbol{y}|}{|\boldsymbol{x}_2 \ldots \boldsymbol{x}_{n+1} \boldsymbol{x}_{n+2}|} \\ \frac{|\boldsymbol{x}_1 \boldsymbol{x}_3 \ldots \boldsymbol{x}_{n+1} \boldsymbol{y}|}{|\boldsymbol{x}_1 \boldsymbol{x}_3 \ldots \boldsymbol{x}_{n+1} \boldsymbol{x}_{n+2}|} \\ \vdots \\ \frac{|\boldsymbol{x}_1 \ldots \boldsymbol{x}_n \boldsymbol{y}|}{|\boldsymbol{x}_1 \ldots \boldsymbol{x}_n \boldsymbol{x}_{n+2}|} \end{bmatrix}. \tag{1}$$

Proof. It is clear that (1) describes a projective transformation, since all the expressions are linear in the coordinates of $\boldsymbol{y}$. It also follows from elementary properties of determinants that (1) maps $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n+2}$ to the standard basis. □

We now turn to Cremona transformations and justify the general formula (5) given in the paper.

Proposition 1.2. *The rational map of $\mathbb{P}^3$ given by*

$$\hat{\boldsymbol{y}} = \boldsymbol{Z} \begin{bmatrix} \frac{1}{|\boldsymbol{y} \boldsymbol{x}_2 \boldsymbol{x}_3 \boldsymbol{x}_4|} \\ \frac{1}{|\boldsymbol{z}_1 \boldsymbol{y} \boldsymbol{z}_3 \boldsymbol{z}_4|} \\ \frac{1}{|\boldsymbol{z}_1 \boldsymbol{z}_2 \boldsymbol{y} \boldsymbol{z}_4|} \\ \frac{1}{|\boldsymbol{z}_1 \boldsymbol{z}_2 \boldsymbol{z}_3 \boldsymbol{y}|} \end{bmatrix}, \tag{2}$$

where $\boldsymbol{Z}$ has columns $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_4$, is a Cremona transformation relative to $\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3, \boldsymbol{z}_4$ (in the sense of Lemma 2.2 from the main part of the paper).

Proof. We need to show that $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_4, \boldsymbol{y}_1, \boldsymbol{y}_2$ are in the same projective configuration as $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_4, \hat{\boldsymbol{y}}_2, \hat{\boldsymbol{y}}_1$. Indeed, a projective transformation relating the two sets of points is given by

$$\boldsymbol{x} \mapsto \boldsymbol{Z} \begin{bmatrix} \frac{|\boldsymbol{x} \boldsymbol{z}_2 \boldsymbol{z}_3 \boldsymbol{z}_4|}{|\boldsymbol{y}_1 \boldsymbol{z}_2 \boldsymbol{z}_3 \boldsymbol{z}_4||\boldsymbol{y}_2 \boldsymbol{z}_2 \boldsymbol{z}_3 \boldsymbol{x}_4|} \\ \frac{|\boldsymbol{z}_1 \boldsymbol{x} \boldsymbol{z}_3 \boldsymbol{z}_4|}{|\boldsymbol{z}_1 \boldsymbol{y}_1 \boldsymbol{z}_3 \boldsymbol{z}_4||\boldsymbol{z}_1 \boldsymbol{y}_2 \boldsymbol{z}_3 \boldsymbol{x}_4|} \\ \frac{|\boldsymbol{z}_1 \boldsymbol{z}_2 \boldsymbol{x} \boldsymbol{z}_4|}{|\boldsymbol{z}_1 \boldsymbol{z}_2 \boldsymbol{y}_1 \boldsymbol{z}_4||\boldsymbol{z}_1 \boldsymbol{z}_2 \boldsymbol{y}_2 \boldsymbol{x}_4|} \\ \frac{|\boldsymbol{z}_1 \boldsymbol{z}_2 \boldsymbol{z}_3 \boldsymbol{x}|}{|\boldsymbol{z}_1 \boldsymbol{z}_2 \boldsymbol{z}_3 \boldsymbol{y}_1||\boldsymbol{z}_1 \boldsymbol{z}_2 \boldsymbol{z}_3 \boldsymbol{y}_2|} \end{bmatrix}, \tag{3}$$

where $\boldsymbol{y}_1, \boldsymbol{y}_2$ are considered fixed. Note that when $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_4$ are basis points then (2) yields the standard Cremona transformation

$$\hat{\boldsymbol{y}} = \left(\frac{1}{y_1}, \frac{1}{y_2}, \frac{1}{y_3}, \frac{1}{y_4}\right)^T, \tag{4}$$

and (3) becomes

$$\boldsymbol{x} \mapsto \left(\frac{x_1}{y_{11}y_{21}}, \frac{x_2}{y_{12}y_{22}}, \frac{x_3}{y_{13}y_{23}}, \frac{x_4}{y_{14}y_{24}}\right)^T, \tag{5}$$

where $\boldsymbol{x} = (x_1, x_2, x_3, x_4)^T$, $\boldsymbol{y}_1 = (y_{11}, y_{12}, y_{13}, y_{14})^T$ and $\boldsymbol{y}_2 = (y_{21}, y_{22}, y_{23}, y_{24})^T$. □

We next prove Proposition 3.1 from the main part of the paper. We recall that the set S in $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^2$ was defined as the set of triples $(\boldsymbol{c}, \boldsymbol{x}, \boldsymbol{u})$ such that $P_{\boldsymbol{c}}(\boldsymbol{x}) = \boldsymbol{u}$ where $P_{\boldsymbol{c}}$ is a reduced camera.

Proposition 3.1. **(1)** *For fixed $\mathbf{c}$ and $\boldsymbol{u}$, the set of points $\boldsymbol{x}$ such that $(\boldsymbol{c}, \boldsymbol{x}, \boldsymbol{u})$ belongs to S is a line with Plücker coordinates*

$$\boldsymbol{\xi} = Q_{\boldsymbol{c}} \boldsymbol{u} \text{ where } Q_{\boldsymbol{c}} = \begin{bmatrix} c_1c_4 & 0 & 0 \\ 0 & c_2c_4 & 0 \\ 0 & 0 & c_3c_4 \\ 0 & -c_2c_3 & c_2c_3 \\ c_1c_3 & 0 & -c_1c_3 \\ -c_1c_2 & c_1c_2 & 0 \end{bmatrix}. \tag{6}$$

(2) *For fixed $\boldsymbol{x}$ and $\boldsymbol{u}$, the set of points $\mathbf{c}$ such that $(\boldsymbol{c}, \boldsymbol{x}, \boldsymbol{u})$ belongs to S is a twisted cubic passing through $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_4$ and $\boldsymbol{x}$.*

Proof. To show (1), we define $\boldsymbol{\xi}_j = \boldsymbol{c} \vee \boldsymbol{z}_j$ $(j = 1, \ldots, 4)$. With our choice of coordinate system, we have

$$\boldsymbol{\xi}_1 = \begin{bmatrix} c_4 \\ 0 \\ 0 \\ 0 \\ c_3 \\ -c_2 \end{bmatrix}, \boldsymbol{\xi}_2 = \begin{bmatrix} 0 \\ c_4 \\ 0 \\ -c_3 \\ 0 \\ c_1 \end{bmatrix}, \boldsymbol{\xi}_3 = \begin{bmatrix} 0 \\ 0 \\ c_4 \\ c_2 \\ -c_1 \\ 0 \end{bmatrix}, \boldsymbol{\xi}_4 = -\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{7}$$

We write $\boldsymbol{\xi} = \rho_1 u_1 \boldsymbol{\xi}_1 + \rho_2 u_2 \boldsymbol{\xi}_2 + \rho_3 u_3 \boldsymbol{\xi}_3$, where the scalars ρ_1, ρ_2, ρ_3 have been chosen so that $\boldsymbol{\xi}_4 = \rho_1 \boldsymbol{\xi}_1 + \rho_2 \boldsymbol{\xi}_2 + \rho_3 \boldsymbol{\xi}_3$. A simple calculation shows that $\rho_1 = -c_1/c_4$, $\rho_2 = -c_2/c_4$, $\rho_3 = -c_3/c_4$, and (6) immediately follows.

For the second statement, we first note that the set S is characterized algebraically by the relation

$$\operatorname{rk} \begin{bmatrix} c_1x_4 - c_4x_1 & u_1c_1 \\ c_2x_4 - c_4x_2 & u_2c_2 \\ c_3x_4 - c_4x_3 & u_3c_3 \end{bmatrix} = 1. \tag{8}$$

This expression follows from

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \sim P_{\boldsymbol{c}}(\boldsymbol{x}) = \begin{bmatrix} x_1/c_1 - x_4/c_4 \\ x_2/c_2 - x_4/c_4 \\ x_3/c_3 - x_4/c_4 \end{bmatrix}, \tag{9}$$

after we clear denominators and eliminate a factor of c_4. We now observe that for fixed $\boldsymbol{x}$ and $\boldsymbol{u}$, the three quadratic equations in $\boldsymbol{c}$ from the minors of (8) define a twisted cubic curve. Indeed, as shown for example in [1, p.14], if L_1, L_2, L_3, M_1, M_2, M_3 are linear forms in c_1, c_2, c_3, c_4, then the projective set defined by

$$\operatorname{rk} \begin{bmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \end{bmatrix} = 1, \tag{10}$$

is a twisted cubic if and only if for any (λ, μ) the three linear forms $\lambda L_i + \mu M_i$ $(i = 1, 2, 3)$ are independent. It follows that (8) is a twisted cubic in $\boldsymbol{x}$ if and only if

$$\begin{bmatrix} -\lambda x_4 + \mu u_1 & 0 & 0 & \lambda x_1 \\ 0 & -\lambda x_4 + \mu u_2 & 0 & \lambda x_2 \\ 0 & 0 & -\lambda x_4 + \mu u_3 & \lambda x_3 \end{bmatrix} \tag{11}$$

has rank three for all λ, μ. This is true if the coordinates u_1, u_2, u_3 are all distinct and not zero, and the coordinates x_1, x_2, x_3 are not zero, which is indeed the case under our genericity assumptions. □

1.1. Reduced 2D trifocal tensors and trilinearities

We begin by presenting some properties of the SFM problem for projections from $\mathbb{P}^2$ to $\mathbb{P}^1$. We represent an analytical projection of this type using a 2×3 real matrix $\boldsymbol{P}$ defined up to scale. As for traditional cameras, the center of this projection is the point of $\mathbb{P}^2$ associated with the null-space of $\boldsymbol{P}$. Note that if $\boldsymbol{P}, \boldsymbol{P}'$ are two projections with distinct centers, then every pair of points $(\boldsymbol{t}, \boldsymbol{t}')$ in $\mathbb{P}^1 \times \mathbb{P}^1$ will be a "correspondence" for $\boldsymbol{P}, \boldsymbol{P}'$, *i.e.*, there will always exist a point $\boldsymbol{x}$ in $\mathbb{P}^2$ such that $\boldsymbol{P}\boldsymbol{x} \sim \boldsymbol{t}$ and $\boldsymbol{P}'\boldsymbol{x} \sim \boldsymbol{t}'$. This follows from the fact that two lines in $\mathbb{P}^2$ always intersect. Thus, the first interesting case of multi-view geometry is for $n = 3$.

Proposition 1.3. *Given three projections $\boldsymbol{P}, \boldsymbol{P}', \boldsymbol{P}''$ with disjoint centers, there exists a $2 \times 2 \times 2$ "trifocal tensor" $\mathcal{T}$ such that $(\boldsymbol{t}, \boldsymbol{t}', \boldsymbol{t}'')$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ correspond if an only if*

$$\mathcal{T}^{ijk} t_i t'_j t''_k = 0, \tag{12}$$

where we use Einstein notation for summation. Entries for $\mathcal{T}$ are given by

$$\mathcal{T}^{ijk} = (-1)^{ijk} |P_{3-i} P'_{3-j} P''_{3-k}|, \;\; i, j, k \in \{1, 2\}, \tag{13}$$

where $|P_i P'_j P''_k|$ denotes the determinant of the 3×3 matrix obtained by stacking the i-th row of $\boldsymbol{P}$, the j-th row of $\boldsymbol{P}'$, and the k-th row of $\boldsymbol{P}''$. Finally, the trifocal tensor $\mathcal{T}$ satisfies the following properties:

1. *Any general $2 \times 2 \times 2$ tensor is a "valid" trifocal tensor.*
2. *Given a general trifocal tensor $\mathcal{T}$, there are* two *(possibly complex) projectively distinct sets of parameters for $\boldsymbol{P}, \boldsymbol{P}', \boldsymbol{P}''$.*

Proof. These facts are shown in [2], but we give a short proof here for completeness. We first note that for any 2×3 projection matrix $\boldsymbol{P}$ there is an associated 3×2 "inverse projection" matrix $\boldsymbol{Q}$ that maps points in $\mathbb{P}^1$ to the corresponding viewing lines in $\mathbb{P}^2$ for $\boldsymbol{P}$ (expressed using dual coordinates). The relation between $\boldsymbol{P}$ and $\boldsymbol{Q}$ is simply

$$\boldsymbol{Q} = \boldsymbol{P}^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{14}$$

Since three lines in $\mathbb{P}^2$ written in dual coordinates as $\boldsymbol{l}, \boldsymbol{l}', \boldsymbol{l}''$ converge if and only if $|\boldsymbol{l}, \boldsymbol{l}', \boldsymbol{l}''| = 0$, we see that $(\boldsymbol{t}, \boldsymbol{t}', \boldsymbol{t}'')$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ form a correspondence for $\boldsymbol{P}, \boldsymbol{P}', \boldsymbol{P}''$ if and only if

$$|\; \boldsymbol{Q}\boldsymbol{t} \quad \boldsymbol{Q}'\boldsymbol{t}' \quad \boldsymbol{Q}''\boldsymbol{t}'' \;| = 0, \tag{15}$$

where $\boldsymbol{Q}, \boldsymbol{Q}', \boldsymbol{Q}''$ denote the inverse projection matrices for $\boldsymbol{P}, \boldsymbol{P}', \boldsymbol{P}''$. The expansion of this determinant, together with (14), immediately yields (12) and (13).

The properties (1) and (2) can be shown computationally. Alternatively, one can explicitly a describe method for reconstructing two projectively distinct projection mappings from a general tensor. We do this below in the case of "reduced" tensors. The fact that every tensor is a valid trifocal tensor can also be argued informally by noting that each projection has 5 degrees of freedom, so after removing projective ambiguity (with 8 parameters) we are left with

$5+5+5-8=7$ which correspond to all $2\times2\times2$ tensors up to scale.[1] □ □

We now consider "reduced" projection mappings from $\mathbb{P}^2$ to $\mathbb{P}^1$. Similarly to the 3D case discussed in the paper, a reduced projection is determined by a center together with 3 fixed points in $\mathbb{P}^2$ in general position. In the following, we will always assume that the three fixed points are basis points for a projective reference frame. This leads to projection matrices of the form

$$\boldsymbol{P_c} = \begin{bmatrix} 1/c_1 & 0 & -1/c_3 \\ 0 & 1/c_2 & -1/c_3 \end{bmatrix} = \begin{bmatrix} \hat{c}_1 & 0 & -\hat{c}_3 \\ 0 & \hat{c}_2 & -\hat{c}_3 \end{bmatrix}, \quad (16)$$

where $\boldsymbol{c} = (c_1, c_2, c_3)^T$ is the center of projection, and we write $\hat{c}_i = 1/c_i$ for convenience. Specializing (13) for three reduced cameras $P_{\boldsymbol{c}}, P_{\boldsymbol{c}'}, P_{\boldsymbol{c}''}$ with $\boldsymbol{c} = (1,1,1)^T$, $\boldsymbol{c}' = (c_1', c_2', c_3')^T$, $\boldsymbol{c}'' = (c_1'', c_2'', c_3'')^T$, yields

$$\begin{array}{ll} \mathcal{T}^{111} = 0, & \mathcal{T}^{112} = \hat{c}_3'\hat{c}_1'' - \hat{c}_2'\hat{c}_1'' \\ \mathcal{T}^{121} = \hat{c}_1'\hat{c}_2'' - \hat{c}_1'\hat{c}_3'', & \mathcal{T}^{122} = \hat{c}_1'\hat{c}_3'' - \hat{c}_3'\hat{c}_1'', \\ \mathcal{T}^{211} = \hat{c}_2'\hat{c}_3'' - \hat{c}_3'\hat{c}_2'', & \mathcal{T}^{212} = \hat{c}_2'\hat{c}_1'' - \hat{c}_2'\hat{c}_3'', \\ \mathcal{T}^{221} = \hat{c}_3'\hat{c}_2'' - \hat{c}_1'\hat{c}_2'', & \mathcal{T}^{222} = 0. \end{array} \quad (17)$$

In addition to $\mathcal{T}^{111} = 0$ and $\mathcal{T}^{222} = 0$, we note that the expressions for the remaining six coefficients always sum to zero. Indeed, there are three "synthetic" constraints arising from the fact that the standard basis points in each image provide by construction three correspondences. However, these three are the *only* constraints that such a tensor must satisfy. In fact, we can give the following simple algorithm for 2D reduced SFM.

1. Change coordinates in each image $\mathbb{P}^1$ using three triples of correspondences to restrict to reduced projection matrices as in (16).
2. Estimate the six non-zero coefficients a, b, c, d, e, f of the reduced trifocal tensor (17), under the condition that they sum to zero, using an arbitrary number of correspondences (at least five) in the new image coordinates.
3. Use (17) to recover $\hat{c}_1', \hat{c}_2', \hat{c}_3'$ and $\hat{c}_1'', \hat{c}_2'', \hat{c}_3''$, and hence $\boldsymbol{c}', \boldsymbol{c}''$ from the six coefficients a, b, c, d, e, f. There are in general *two* solutions, which can be computed as follows. Writing $\rho_{ij} = \hat{c}_i'\hat{c}_j''$, we have that (17) yields the following linear relation

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_{12} \\ \rho_{13} \\ \rho_{21} \\ \rho_{23} \\ \rho_{31} \\ \rho_{32} \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}. \quad (18)$$

The matrix on the left has rank 5, with a null-space generated by $(1,1,1,1,1,1)^T$. This means that we may write $\rho_{ij} = e_{ij} + t$ where $(e_{12}, e_{13}, e_{21}, e_{23}, e_{31}, e_{32})$ is any vector that satisfies (18) and t is unknown. However, we may solve for t using the fact that ρ_{ij} must satisfy

$$\rho_{12}\rho_{23}\rho_{31} - \rho_{13}\rho_{21}\rho_{32} = 0. \quad (19)$$

This yields a constraint on t, which is actually quadratic rather than cubic, since the cubic term in t cancels out from the two summands. Given a valid set of ρ_{ij}, it is straightforward to recover $\hat{c}_i'$ and $\hat{c}_j''$. Indeed, it is sufficient for example to set

$$\begin{array}{l} (\hat{c}_1', \hat{c}_2', \hat{c}_3') = (\rho_{12}/\rho_{32}, \rho_{21}/\rho_{31}, 1), \\ (\hat{c}_1'', \hat{c}_2'', \hat{c}_3'') = (\rho_{21}/\rho_{23}, \rho_{12}/\rho_{13}, 1). \end{array} \quad (20)$$

Note that this procedure is very similar to the method used in the proof of Proposition 4.3 in the paper. Indeed, the trilinearities T_1, T_2, T_3, T_4 are closely related to the 2D reduced trifocal tensor, and we will now spell out this relation in detail. We first recall the following fact from [3] (see also Proposition 1.1 in the paper).

Proposition 1.4. *Three image points $\boldsymbol{u}, \boldsymbol{u}', \boldsymbol{u}''$ satisfy the trilinearity T_i if and only if the three associated viewing lines $\boldsymbol{\xi}, \boldsymbol{\xi}', \boldsymbol{\xi}''$ admit a common transveral through the basis point $\boldsymbol{z}_i$.*

The geometric condition from Proposition 1.4 can be expressed by considering an arbitrary projection $P_{\boldsymbol{z}_i}$ with center $\boldsymbol{z}_i$, and imposing that the projections $\boldsymbol{l}, \boldsymbol{l}', \boldsymbol{l}''$ of $\boldsymbol{\xi}, \boldsymbol{\xi}', \boldsymbol{\xi}''$ under $P_{\boldsymbol{z}_i}$ are lines in $\mathbb{P}^2$ that converge at a point. This is why 2D trifocal tensors come into play.

In what follows, we write $\boldsymbol{c}_r', \boldsymbol{c}_r''$ for the points in $\mathbb{P}^2$ obtained from $\boldsymbol{c}', \boldsymbol{c}''$ by excluding the r-th coordinate (*e.g.*, $\boldsymbol{c}_2' = (c_1', c_3', c_4')^T$).

Proposition 1.5. *The trilinearity T_r ($r = 1,2,3,4$) applied to $\boldsymbol{u}, \boldsymbol{u}', \boldsymbol{u}''$ can be written as*

$$\mathcal{T}_r^{ijk}(\boldsymbol{M}_r\boldsymbol{u})_i(\boldsymbol{M}_r\boldsymbol{u}')_j(\boldsymbol{M}_r\boldsymbol{u}'')_k = 0, \quad (21)$$

where $\mathcal{T}_r^{ijk}$ is the reduced 2D trifocal tensor associated with $(1,1,1)^T$, $\boldsymbol{c}_r'$, $\boldsymbol{c}_r''$ (see (17) for the coefficients) and

$$\begin{array}{ll} \boldsymbol{M}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & \boldsymbol{M}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ \boldsymbol{M}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, & \boldsymbol{M}_4 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}. \end{array} \quad (22)$$

Proof. This property can be verified computationally with a computer algebra system using the expressions for T_i (given in Proposition 8 of the paper), and for the reduced trifocal tensor (17). A geometric justification based on Proposition 1.4 is as follows.

[1] To make this argument more precise, one needs to observe that projective transformations act freely on triples of cameras (*i.e.*, no projective transformation of $\mathbb{P}^2$ fixes $\boldsymbol{P}, \boldsymbol{P}', \boldsymbol{P}''$ simultaneously).

Let us focus on T_1, for notational simplicity. The argument is identical for the other trilinearities. We consider the following simple projection with pinhole $\boldsymbol{z}_1$

$$\boldsymbol{P}_{\boldsymbol{z}_1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

Any triple of image points $\boldsymbol{u}, \boldsymbol{u}', \boldsymbol{u}''$ (in distinct images) determines three lines $\boldsymbol{l}, \boldsymbol{l}', \boldsymbol{l}''$ in the image plane of $P_{\boldsymbol{z}_1}$, which contain the points $(1,1,1)^T$, $\boldsymbol{c}_1' = (c_2', c_3', c_4')^T$, $\boldsymbol{c}_1'' = (c_2'', c_3'', c_4'')^T$. These three points are in fact the images of $\boldsymbol{c}, \boldsymbol{c}', \boldsymbol{c}''$ under $P_{\boldsymbol{z}_1}$ (*i.e.*, they are three epipoles). We can parameterize the three bundles through $\boldsymbol{c}_1, \boldsymbol{c}_1', \boldsymbol{c}_1''$ using the reference frame induced by the three points $P_{\boldsymbol{z}_1}(\boldsymbol{z}_2)$, $P_{\boldsymbol{z}_1}(\boldsymbol{z}_3)$, $P_{\boldsymbol{z}_1}(\boldsymbol{z}_4)$. This way, converging triplets $\boldsymbol{l}, \boldsymbol{l}', \boldsymbol{l}''$ are described by a reduced 2D trifocal tensor.

Finally, the coordinates in $\mathbb{P}^1$ of the lines associated with $\boldsymbol{u}, \boldsymbol{u}', \boldsymbol{u}''$ are given simply by $\boldsymbol{M}_1\boldsymbol{u}$, $\boldsymbol{M}_1\boldsymbol{u}'$, $\boldsymbol{M}_1\boldsymbol{u}''$. This is because $\boldsymbol{M}_1$ is the unique projection matrix that maps $(0,1,0)^T$, $(0,0,1)^T$, $(1,1,1)^T$ (which are images of $\boldsymbol{z}_2, \boldsymbol{z}_3, \boldsymbol{z}_4$ for all three cameras) to the standard basis of $\mathbb{P}^1$.

In conclusion, a triplet $\boldsymbol{u}, \boldsymbol{u}', \boldsymbol{u}''$ satisfies T_1 if and only if the associated lines $\boldsymbol{l}, \boldsymbol{l}', \boldsymbol{l}''$ in the image plane of $P_{\boldsymbol{z}_1}$ converge, and this in turn is equivalent to the fact that $\boldsymbol{M}_1\boldsymbol{u}$, $\boldsymbol{M}_1\boldsymbol{u}'$, $\boldsymbol{M}_1\boldsymbol{u}''$ are three points in $\mathbb{P}^1$ which satisfy the constraint from the reduced 2D trifocal tensor. □

Expanding (21) for $r = 1,2,3$ we obtain the expressions for T_1, T_2, T_3 given in equation (10) of the main part of the paper. For completeness, we include here the explicit expanded form of all the trilinearities:

$$\begin{aligned} T_1 =\, & u_2u_2'u_3''(-\hat{c}_3'\hat{c}_2'' + \hat{c}_4'\hat{c}_2'') + u_2u_3'u_2''(\hat{c}_2'\hat{c}_3'' - \hat{c}_2'\hat{c}_4'') + \\ & u_2u_3'u_3''(-\hat{c}_4'\hat{c}_2'' + \hat{c}_2'\hat{c}_4'') + u_3u_2'u_2''(-\hat{c}_4'\hat{c}_3'' + \hat{c}_3'\hat{c}_4'') + \\ & u_3u_2'u_3''(\hat{c}_3'\hat{c}_2'' - \hat{c}_3'\hat{c}_4'') + u_3u_3'u_2''(-\hat{c}_2'\hat{c}_3'' + \hat{c}_4'\hat{c}_3''), \end{aligned}$$

$$\begin{aligned} T_2 =\, & u_1u_1'u_3''(-\hat{c}_3'\hat{c}_1'' + \hat{c}_4'\hat{c}_1'') + u_1u_3'u_1''(\hat{c}_1'\hat{c}_3'' - \hat{c}_1'\hat{c}_4'') + \\ & u_1u_3'u_3''(-\hat{c}_4'\hat{c}_1'' + \hat{c}_1'\hat{c}_4'') + u_3u_1'u_1''(-\hat{c}_4'\hat{c}_3'' + \hat{c}_3'\hat{c}_4'') + \\ & u_3u_1'u_3''(\hat{c}_3'\hat{c}_1'' - \hat{c}_3'\hat{c}_4'') + u_3u_3'u_1''(-\hat{c}_1'\hat{c}_3'' + \hat{c}_4'\hat{c}_3''), \end{aligned}$$

$$\begin{aligned} T_3 =\, & u_1u_1'u_2''(-\hat{c}_2'\hat{c}_1'' + \hat{c}_4'\hat{c}_1'') + u_1u_2'u_1''(\hat{c}_1'\hat{c}_2'' - \hat{c}_1'\hat{c}_4'') + \\ & u_1u_2'u_2''(-\hat{c}_4'\hat{c}_1'' + \hat{c}_1'\hat{c}_4'') + u_2u_1'u_1''(-\hat{c}_4'\hat{c}_2'' + \hat{c}_2'\hat{c}_4'') + \\ & u_2u_1'u_2''(\hat{c}_2'\hat{c}_1'' - \hat{c}_2'\hat{c}_4'') + u_2u_2'u_1''(-\hat{c}_1'\hat{c}_2'' + \hat{c}_4'\hat{c}_2''), \end{aligned}$$

$$\begin{aligned} T_4 =\, & u_1u_1'u_2''(-\hat{c}_2'\hat{c}_1'' + \hat{c}_3'\hat{c}_1'') + u_1u_1'u_3''(\hat{c}_2'\hat{c}_1'' - \hat{c}_3'\hat{c}_1'') + \\ & u_1u_2'u_1''(\hat{c}_1'\hat{c}_2'' - \hat{c}_1'\hat{c}_3'') + u_1u_2'u_2''(-\hat{c}_3'\hat{c}_1'' + \hat{c}_1'\hat{c}_3'') + \\ & u_1u_2'u_3''(\hat{c}_3'\hat{c}_1'' - \hat{c}_1'\hat{c}_2'') + u_1u_3'u_1''(-\hat{c}_1'\hat{c}_2'' + \hat{c}_1'\hat{c}_3'') + \\ & u_1u_3'u_2''(\hat{c}_2'\hat{c}_1'' - \hat{c}_1'\hat{c}_3'') + u_1u_3'u_3''(-\hat{c}_2'\hat{c}_1'' + \hat{c}_1'\hat{c}_2'') + \\ & u_2u_1'u_1''(-\hat{c}_3'\hat{c}_2'' + \hat{c}_2'\hat{c}_3'') + u_2u_1'u_2''(\hat{c}_2'\hat{c}_1'' - \hat{c}_2'\hat{c}_3'') + \\ & u_2u_1'u_3''(-\hat{c}_2'\hat{c}_1'' + \hat{c}_3'\hat{c}_2'') + u_2u_2'u_1''(-\hat{c}_1'\hat{c}_2'' + \hat{c}_3'\hat{c}_2'') + \\ & u_2u_2'u_3''(\hat{c}_1'\hat{c}_2'' - \hat{c}_3'\hat{c}_2'') + u_2u_3'u_1''(\hat{c}_1'\hat{c}_2'' - \hat{c}_2'\hat{c}_3'') + \\ & u_2u_3'u_2''(-\hat{c}_2'\hat{c}_1'' + \hat{c}_2'\hat{c}_3'') + u_2u_3'u_3''(\hat{c}_2'\hat{c}_1'' - \hat{c}_1'\hat{c}_2'') + \\ & u_3u_1'u_1''(\hat{c}_3'\hat{c}_2'' - \hat{c}_2'\hat{c}_3'') + u_3u_1'u_2''(-\hat{c}_3'\hat{c}_1'' + \hat{c}_2'\hat{c}_3'') + \\ & u_3u_1'u_3''(\hat{c}_3'\hat{c}_1'' - \hat{c}_3'\hat{c}_2'') + u_3u_2'u_1''(-\hat{c}_3'\hat{c}_2'' + \hat{c}_1'\hat{c}_3'') + \\ & u_3u_2'u_2''(\hat{c}_3'\hat{c}_1'' - \hat{c}_1'\hat{c}_3'') + u_3u_2'u_3''(-\hat{c}_3'\hat{c}_1'' + \hat{c}_3'\hat{c}_2'') + \\ & u_3u_3'u_1''(-\hat{c}_1'\hat{c}_3'' + \hat{c}_2'\hat{c}_3'') + u_3u_3'u_2''(\hat{c}_1'\hat{c}_3'' - \hat{c}_2'\hat{c}_3''). \end{aligned} \tag{24}$$

We can also use Proposition 1.5 to prove theoretical properties of all four trilinearities. The following statement is equivalent to Proposition 4.3 in the main part of the paper.

Proposition 1.6. *The internal constraints of each trilinearity T_r are* linear. *More precisely, the coefficients in $\mathbb{R}^{27}$ that are entries of T_i for some choice of $\boldsymbol{c}', \boldsymbol{c}''$ form a* linear space, *of dimension five. Moreover, the coefficients of T_r characterize $\boldsymbol{c}_r'$ and $\boldsymbol{c}_r''$ up to a two-fold ambiguity.*

Proof. It follows from (21) that the coefficients of T_r are images of the coefficients of $\mathcal{T}_r^{ijk}$ under an injective linear map (note that this map is completely described by the four matrices (22)). Since the entries of a reduced 2D trifocal are only constrained by three linear conditions, the coefficients of T_r form linear spaces of dimension five. Moreover, the knowledge of the T_r is equivalent to that of $\mathcal{T}_r^{ijk}$, which means that $\boldsymbol{c}_r'$ and $\boldsymbol{c}_r''$ are determined up to a two-fold ambiguity. □

Finally, we provide some details on the proof of Proposition 4.2 in the paper.

Proposition 4.2 *A vector $\boldsymbol{d} = (d_{ij})$ in $\mathbb{R}^{12}$ with no zero entries can be written as $d_{ij} = a_ib_j$ for some vectors $\boldsymbol{a} = (a_1,a_2,a_3,a_4)^T, \boldsymbol{b} = (b_1,b_2,b_3,b_4)^T$ in $\mathbb{R}^3$ if and only if $d_{ij}d_{kl} = d_{il}d_{kj}$ holds for all permutations (i,j,k,l) of $(1,2,3,4)$.*

Proof. A factorization of $\boldsymbol{d} = (d_{ij})$ exists if and only if there is a rank-1 completion of

$$\begin{pmatrix} * & d_{12} & d_{13} & d_{14} \\ d_{21} & * & d_{23} & d_{24} \\ d_{31} & d_{32} & * & d_{34} \\ d_{41} & d_{42} & d_{43} & * \end{pmatrix}. \tag{25}$$

Indeed, a completion corresponds to $\boldsymbol{a}^T\boldsymbol{b} \in \mathbb{R}^{4\times 4}$. It is clear that the constraints $d_{ij}d_{kl} = d_{il}d_{kj}$, which correspond to certain 2×2 minors, are necessary. Conversely, it is always possible to deduce the diagonal elements, for example:

$$d_{11} = \frac{d_{13}d_{21}}{d_{23}} = \frac{d_{14}d_{21}}{d_{24}} = \frac{d_{12}d_{31}}{d_{32}} = \frac{d_{14}d_{31}}{d_{34}}. \tag{26}$$

The constraints $d_{ij}d_{kl} = d_{il}d_{kj}$ guarantee equality among all these expressions. Indeed, we have that for example $d_{12}d_{23}d_{31} = d_{21}d_{32}d_{13}$ because

$$\frac{d_{12}}{d_{13}}\frac{d_{23}}{d_{21}}\frac{d_{31}}{d_{32}} = \frac{d_{42}}{d_{43}}\frac{d_{43}}{d_{41}}\frac{d_{31}}{d_{32}} = \frac{d_{42}d_{31}}{d_{43}d_{32}} = 1. \tag{27}$$

This shows that if the quadratic constraints hold, we can solve for the diagonal elements in (25), and the resulting matrix will have rank one. □

2. Synthetic data experiment

For completeness, we show in Figure 1 the *mean* value of the mean reprojection and reconstruction errors for the synthetic data used in Section 5.2 of our submission. Recall that the quality of the reconstruction was evaluated in that section by measuring how well it predicts the reprojection of the remaining points in the dataset as well as their 3D reconstruction, once again registered to the ground truth through a homography. We have used the same data as in Section 5.2 to construct the curves shown in Figure 1. They show the mean values of the mean reprojection and reconstruction errors, given respectively in pixel and mm, for 40 random choices of the 7 point correspondences and different values of the standard deviation σ (in pixel units) of Gaussian noise added to the image coordinates. As noted in our submission, Figure 1 shows that both the linear trifocal tensor estimation are occasionally thrown completely off course for "bad" choices of the 7 correspondences, without a clear winner in this case.

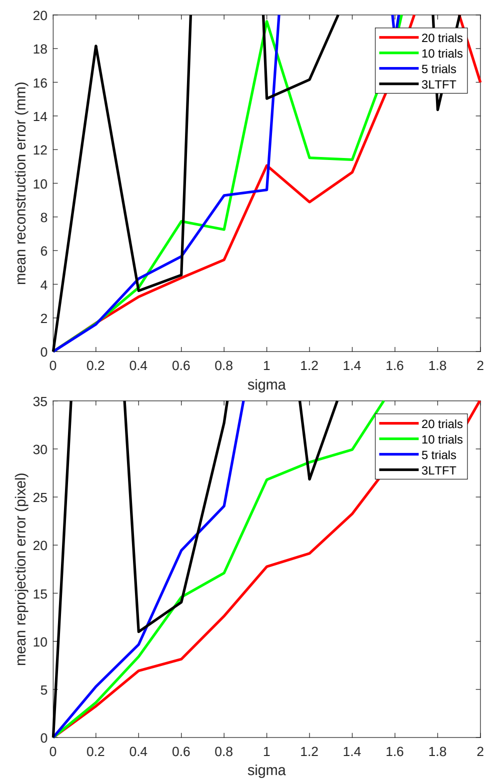

Figure 1. Experiments with synthetic data using the mean of the mean reprojection and reconstruction errors instead of their median. Compare to Figure 8 in our submission.

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