

A sparse resultant based method for efficient minimal solvers

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Abstract

Many computer vision applications require robust and efficient estimation of camera geometry. The robust estimation is usually based on solving camera geometry problems from a minimal number of input data measurements, i.e. solving minimal problems in a RANSAC framework. Minimal problems often result in complex systems of polynomial equations. Many state-of-the-art efficient polynomial solvers to these problems are based on Gröbner bases and the action-matrix method that has been automated and highly optimized in recent years. In this paper we study an alternative algebraic method for solving systems of polynomial equations, i.e., the sparse resultant-based method and propose a novel approach to convert the resultant constraint to an eigenvalue problem. This technique can significantly improve the efficiency and stability of existing resultant-based solvers. We applied our new resultant-based method to a large variety of computer vision problems and show that for most of the considered problems, the new method leads to solvers that are the same size as the the best available Gröbner basis solvers and of similar accuracy. For some problems the new sparse-resultant based method leads to even smaller and more stable solvers than the state-of-the-art Gröbner basis solvers. Our new method can be fully automated and incorporated into existing tools for automatic generation of efficient polynomial solvers and as such it represents a competitive alternative to popular Gröbner basis methods for minimal problems in computer vision.

1. Introduction

Computing camera geometry is one of the most important tasks in computer vision [17] with many applications e.g. in structure from motion [39], visual navigation [38], large scale 3D reconstruction [19] and image localization [37].

The robust estimation of camera geometry is usually based on solving so-called minimal problems [35, 24, 23], i.e. problems that are solved from minimal samples of input data, inside a RANSAC framework [14, 9, 36]. Since the camera geometry estimation has to be performed many

times in RANSAC [14], fast solvers to minimal problems are of high importance. Minimal problems often result in complex systems of polynomial equations in several variables. A popular approach for solving minimal problems is to design procedures that can efficiently solve only a special class of systems of equations, e.g. systems resulting from the 5-pt relative pose problem [35], and move as much computation as possible from the “online” stage of solving equations to an earlier pre-processing “offline” stage.

Most of the state-of-the-art specific minimal solvers are based on Gröbner bases and the action-matrix method [10]. The Gröbner basis method was popularized in computer vision by Stewenius [40]. The first efficient Gröbner basis solvers were mostly handcrafted [41, 42] and sometimes very unstable [43]. However, in the last 15 years much effort has been put into making the process of constructing the solvers more automatic [24, 29, 30] and the solvers stable [6, 7] and more efficient [29, 30, 28, 5, 32]. There are now powerful tools available for the automatic generation of efficient Gröbner basis solvers [24, 29].

While the Gröbner basis method for generating efficient minimal solvers was thoroughly studied in computer vision and all recently generated Gröbner basis solvers are highly optimized in terms of efficiency and stability, little attention has been paid to an alternative algebraic method for solving systems of polynomial equations, i.e. the resultant-based method. The resultant-based method was manually applied to several computer vision problems [25, 16, 16, 20, 23, 25]. However in contrast to the Gröbner basis method, there is no general method for automatically generating efficient resultant-based minimal solvers. The most promising results in this direction were proposed by Emiris [12] and Heikkilä [18], where methods based on sparse resultants were proposed and applied to camera geometry problems. While these methods can be extended for general minimal problems that appear in computer vision and can be automated, they usually lead (due to linearizations) to larger and less efficient solvers than Gröbner basis solvers.

In this paper, we propose a novel approach to generating minimal solvers using sparse resultants, which is based on adding an extra equation of a special form to the in-

put system. Our algorithm is inspired by the ideas explored in [18, 12], but thanks to the special form of added equation and by solving the resultant as a small eigenvalue problem, in contrast to a polynomial eigenvalue problem in [18], the new approach achieves significant improvements over [18, 12] in terms of efficiency of the generated solvers. Specifically our contributions include,

- A novel sparse resultant-based approach to generating polynomial solvers based on adding an extra equation of a special form and transforming the resultant matrix constraint to a regular eigenvalue problem.
- Two procedures to reduce the size of resultant matrix that lead to faster solvers than the best available state-of-the-art solvers for some minimal problems.
- A general method for automatic generation of efficient resultant-based polynomial solvers for many important minimal problems that achieves competitive performance in terms of speed and stability with respect to the best available state-of-the-art solvers generated by highly optimized Gröbner basis techniques [29, 32]. The automatic generator of resultant-based solvers will be made publicly available at [45].

2. Theoretical background and related work

In this paper we use notation and basic concepts from the book by Cox *et al.* [10]. Our objective is to solve m polynomial equations,

$$\{f_1(x_1, \dots, x_n) = 0, \dots, f_m(x_1, \dots, x_n) = 0\} \quad (1)$$

in n unknowns, $X = \{x_1, \dots, x_n\}$. Let $\mathbb{C}[X]$ denote the set of all polynomials in unknowns X with coefficients in \mathbb{C} . The ideal $I = \langle f_1, \dots, f_m \rangle \subset \mathbb{C}[X]$ is the set of all polynomial combinations of our generators f_1, \dots, f_m . The set V of all solutions of the system (1) is called the affine variety. Each polynomial $f \in I$ vanishes on the solutions of (1). Here we assume that the ideal I generates a zero dimensional variety, *i.e.* the system (1) has a finite number of solutions. Using the ideal I we can define the quotient ring $A = \mathbb{C}[X]/I$ which is the set of equivalence classes over $\mathbb{C}[X]$ defined by the relation $a \sim b \iff (a - b) \in I$. If I has a zero-dimensional variety then the quotient ring $A = \mathbb{C}[X]/I$ is a finite-dimensional vector space over \mathbb{C} . For an ideal I there exist special sets of generators called Gröbner bases which have the nice property that the remainder after division is unique. Using a Gröbner basis we can define a linear basis for the quotient ring $A = \mathbb{C}[X]/I$.

2.1. Gröbner Basis method

Gröbner bases can be used to solve our system of polynomial equations (1). One of the popular approaches for solving systems of equations using Gröbner bases is the multiplication matrix method, known also as the action matrix

method [10, 44]. This method was recently used to efficiently solve many of the minimal problems in computer vision [23, 24, 29, 32]. The goal of this method is to transform the problem of finding the solutions to (1) to a problem of eigendecomposition of a special multiplication matrix [11]. Let us consider the mapping $T_f : A \rightarrow A$ of the multiplication by a polynomial $f \in \mathbb{C}[X]$. T_f is a linear mapping for which $T_f = T_g$ iff $f - g \in I$. In our case A is a finite-dimensional vector space over \mathbb{C} and therefore we can represent T_f by its matrix with respect to some linear basis B of A . For a basis $B = ([b_1], \dots, [b_k])$ consisting of k monomials, T_f can be represented by $k \times k$ multiplication (action) matrix $M_f := (m_{ij})$ such that $T_f([b_j]) = [fb_j] = \sum_{i=1}^k m_{ij}[b_i]$. It can be shown [11] that $\lambda \in \mathbb{C}$ is an eigenvalue of the matrix M_f iff λ is a value of the function f on the variety V . In other words, if f is *e.g.* x_n then the eigenvalues of M_f are the x_n -coordinates of the solutions of (1). The solutions to the remaining variables can be obtained from the eigenvectors of M_f . This means that after finding the multiplication matrix M_f , we can recover the solutions by solving the eigendecomposition of M_f for which efficient algorithms exist. Moreover, if the ideal I is a radical ideal, *i.e.* $I = \sqrt{I}$, [11], then k is equal to the number of solutions to the system (1). Therefore, Gröbner basis methods usually solve an eigenvalue problem of a size that is equivalent to the number of solutions of the problem. For more details and proofs we refer the reader to [10].

The coefficients of the multiplication matrix M_f are polynomial combinations of coefficients of the input polynomials (1). For computer vision problems these polynomial combinations are often found “offline“ in a pre-processing step. In this step, a so-called *elimination template* is generated, which is actually an expanded set of equations constructed by multiplying original equations with different monomials. This template matrix is constructed such that after filling it with coefficients from the input equations and performing Gauss-Jordan(G-J) elimination of this matrix, the coefficients of the multiplication matrix M_f can be obtained from this eliminated template matrix.

The first automatic approach for generating elimination templates and Gröbner basis solvers was presented in [24]. Recently an improvement to the automatic generator [24] was proposed in [29] to exploit the inherent relations between the input polynomial equations and it results in more efficient solvers than [24]. The automatic method from [29] was later extended by a method for dealing with saturated ideals [30] and a method for detecting symmetries in polynomial systems [28].

In general, the answer to the question “What is the smallest elimination template for a given problem?” is not known. In [32] the authors showed that the method [29], which is based on the grevlex ordering of monomials and the so-called standard bases of the quotient ring A is not

optimal in terms of template sizes. The authors of [29] proposed two methods for generating smaller elimination templates. The first is based on enumerating and testing all Gröbner bases w.r.t. different monomial orderings, i.e., the so-called Gröbner fan. By generating solvers w.r.t. all these Gröbner bases and using standard bases of the quotient ring A , smaller solvers were obtained for many problems. The second method goes “beyond Gröbner bases” and it uses a manually designed heuristic sampling scheme for generating “non-standard” monomial bases B of $A = \mathbb{C}[X]/I$. This heuristic leads to more efficient solvers than the Gröbner fan method in many cases. While the Gröbner fan method will provably generate at least as efficient solvers as the grevlex-based method from [29], no proof can be in general given for the “heuristic-based” method. The proposed heuristic sampling scheme uses only empirical observations on which basis monomials will likely result in small templates and it samples a fixed number (1000 in the paper) of candidate bases consisting of these monomials. Even though, e.g. the standard grevlex monomial basis will most likely be sampled during the sampling, it is in general not clear how large templates it will generate for a particular problem. The results will also depend on the number of bases tested inside the heuristic.

2.2. Sparse Resultants

An alternate approach towards solving polynomial equations is that of using resultants. Simply put, a resultant is an irreducible polynomial constraining coefficients of a set of $n + 1$ polynomials, $F = \{f_1(x_1, \dots, x_n), \dots, f_{n+1}(x_1, \dots, x_n)\}$ in n variables to have a non-trivial solution. One can refer to Cox *et al.* [10] for a more formal theory on resultants. We have $n + 1$ equations in n variables because resultants were initially developed to determine whether a system of polynomial equations has a common root or not. If a coefficient of monomial \mathbf{x}^α in the i^{th} polynomial of F is denoted as $u_{i,\alpha}$ the resultant is a polynomial $\text{Res}([u_{i,\alpha}])$ with $u_{i,\alpha}$ as variables.

Using this terminology, the basic idea for a resultant based method is to expand F to a set of linearly independent polynomials which can be linearised as $M([u_{i,\alpha}])\mathbf{b}$, where \mathbf{b} is a vector of monomials of form \mathbf{x}^α and $M([u_{i,\alpha}])$ has to be a square matrix that has full rank for generic values of $u_{i,\alpha}$, i.e. $\det M([u_{i,\alpha}]) \neq 0$. The determinant of the matrix $M([u_{i,\alpha}])$ is a non-trivial multiple of the resultant $\text{Res}([u_{i,\alpha}])$ [10]. Thus $\det M([u_{i,\alpha}])$ must vanish, if the resultant vanishes, i.e. $\text{Res}([u_{i,\alpha}]) = 0 \implies \det M([u_{i,\alpha}]) = 0$. It is known that $\text{Res}([u_{i,\alpha}])$ vanishes iff the polynomial system F has a solution [10]. This gives us the necessary condition for the existence of roots of $F = 0$. Hence the equation $\det M([u_{i,\alpha}]) = 0$ gives us those values of $u_{i,\alpha}$ such that $F = 0$ have a common root.

Resultants can be used to solve n polynomial equations

in n unknowns. The most common approach used for this purpose is to *hide a variable* by considering it as a constant. By hiding, say x_n , we obtain n polynomials in $n-1$ variables, so we can use the concept of resultants and compute $\text{Res}([u_{i,\alpha}], x_n)$ which now becomes a function of $u_{i,\alpha}$ as well as x_n . Algorithms based on hiding a variable attempt to expand F to a linearly independent set of polynomials that can be re-written in a matrix form as

$$M([u_{i,m}], x_n)\mathbf{b} = 0, \quad (2)$$

where $M([u_{i,\alpha}], x_n)$ is a square matrix whose elements are polynomials in x_n and coefficients $u_{i,\alpha}$ and \mathbf{b} is the vector of monomials in x_1, \dots, x_{n-1} . For simplicity we will denote the matrix $M([u_{i,\alpha}], x_n)$ as $M(x_n)$ in the rest of this section. Here we actually estimate a multiple of the actual resultant via the determinant of the matrix $M(x_n)$ in (2). This resultant is known as a *hidden variable resultant* and it is a polynomial in x_n whose roots are the x_n -coordinates of the solutions of the system of polynomial equations. For theoretical details and proofs see [10]. Such a hidden variable approach has been used in the past to solve various minimal problems [16, 20, 23, 25].

The most common way to solve the original system of polynomial equations is to transform (2) to a polynomial eigenvalue problem (PEP) [11] that transforms (2) as

$$(M_0 + M_1 x_n + \dots + M_l x_n^l)\mathbf{b} = \mathbf{0}, \quad (3)$$

where l is the degree of the matrix $M(x_n)$ in the hidden variable x_n and matrices M_0, \dots, M_l are matrices that depend only on the coefficients $u_{i,\alpha}$ of the original system of polynomials. The PEP (3) can be easily converted to a generalized eigenvalue problem (GEP):

$$\mathbf{A}\mathbf{y} = x_n\mathbf{B}\mathbf{y}, \quad (4)$$

and solved using standard efficient eigenvalue algorithms [25]. Basically, the eigenvalues give us the solution to x_n and the rest of the variables can be solved from the corresponding eigenvectors, \mathbf{y} [10]. But this transformation to a GEP relaxes the original problem of finding the solutions to our input system and computes eigenvectors that do not satisfy the monomial dependencies induced by the monomial vector \mathbf{b} . And many times it also introduces extra parasitic (zero) eigenvalues leading to slower polynomial solvers.

Alternately, we can add a new polynomial

$$f_{n+1} = u_0 + u_1 x_1 + \dots + u_n x_n \quad (5)$$

to F and compute a so called *u-resultant* [10] by hiding u_0, \dots, u_n . In general random values are assigned to u_1, \dots, u_n . The *u-resultant* matrix is computed from these $n + 1$ polynomials in n variables in a way similar to the one explored above. For more details one can refer to [10].

For sparse polynomial systems it is possible to obtain more compact resultants using specialized algorithms. Such resultants are commonly referred to as *Sparse Resultants*. A sparse resultant would mostly lead to a more compact matrix $M(x_n)$ and hence a smaller eigendecomposition problem. Emiris *et al.* [13, 8] have proposed a generalised approach for computing sparse resultants using mixed-subdivision of polytopes. Based on [13, 8] Emiris proposed a method for generating a resultant-based solver for sparse systems of polynomial equations, that was divided in “offline” and “online” computations. The resulting solvers were based either on the hidden-variable trick (2) or the u -resultant of the general form (5). As such the resulting solvers were usually quite large and not very efficient. More recently Heikkilä [18] have proposed an improved approach to test and extract smaller $M(x_n)$. This method transforms (2) to a GEP (4) and solves for eigenvalues and eigenvectors to compute solutions to unknowns. The methods [8, 12, 13, 18] suffer from the drawback that they require the input system to have as many polynomials as unknowns to be able to compute a resultant. Additionally, the algorithm [18] suffers from other drawbacks and can not be directly applied to most of the minimal problems. These drawbacks can be overcome, as we describe in the supplementary material. However, even with our proposed improvements the resultant-based method [18], which is based on hiding one of the input variables in the coefficient field, would result in a GEP with unwanted eigenvalues and in turn unwanted solutions to original system (1). This leads to slower solvers for most of the studied minimal problems.

Therefore, we investigate an alternate approach where instead of hiding one of the input variables [12, 18] or using u -resultant of a general form (5) [12], we introduce an extra variable λ and a new polynomial of a special form, i.e., $x_i - \lambda$. The augmented polynomial system is solved by hiding λ and reducing a constraint similar to (2) into a regular eigenvalue problem that leads to smaller solvers than [12, 18]. Next section lays the theoretical foundation of our approach and outlines the algorithm along with the steps for computing a sparse resultant matrix $M(\lambda)$.

3. Sparse resultants using an extra equation

We start with a set of m polynomials from (1) in n variables x_1, \dots, x_n to be solved. Introducing an extra variable λ we define $\mathbf{x}' = [x_1, \dots, x_n, \lambda]$ and an extra polynomial $f_{m+1}(\mathbf{x}') = x_i - \lambda$. Using this, we propose an algorithm inspired by [18] and [12] to solve the following augmented polynomial system for \mathbf{x}' ,

$$f_1(\mathbf{x}') = 0, \dots, f_m(\mathbf{x}') = 0, f_{m+1}(\mathbf{x}') = 0. \quad (6)$$

Our idea is to compute its sparse resultant matrix $M(= M(\lambda))$ by hiding λ in a way that allows us to solve (6) by reducing its linearization (similar to (2)) to an eigenvalue problem.

3.1. Sparse resultant and eigenvalue problem

Our algorithm computes the monomial multiples of the polynomials in (6) in the form of a set $T = \{T_1, \dots, T_m, T_{m+1}\}$ where each T_i denotes the set of monomials to be multiplied by $f_i(\mathbf{x}')$. We may order monomials in each T_i to obtain a vector form, $\mathbf{T}_i = \text{vec}(T_i)$ and stack these vectors as $\mathbf{T} = [\mathbf{T}_1, \dots, \mathbf{T}_m, \mathbf{T}_{m+1}]$. The set of all monomials present in the resulting extended set of polynomials $\{\mathbf{x}^{\alpha_i} f_i(\mathbf{x}'), \forall \mathbf{x}^{\alpha_i} \in T_i, i = 1, \dots, m+1\}$ is called the monomial basis and is denoted as $B = \{\mathbf{x}^\alpha \mid \alpha \in \mathbb{Z}_{\geq 0}^n\}$. The vector form of B w.r.t. some monomial ordering is denoted as \mathbf{b} . Then the extended set of polynomials can be written in a matrix form,

$$\mathbf{M} \mathbf{b} = 0, \quad (7)$$

The coefficient matrix \mathbf{M} is a function of λ as well as the coefficients of input polynomials (6). Let $\varepsilon = |B|$. Then by construction [18] \mathbf{M} is a tall matrix with $p \geq \varepsilon$ rows. We can remove extra rows and form an invertible square matrix which is the sparse resultant matrix mentioned in previous section. While Heikkilä [18] solve a problem similar to (7) as a GEP, we exploit the structure of newly added polynomial $f_{m+1}(\mathbf{x}')$ and propose a block partition of \mathbf{M} to reduce the matrix equation of (7) to a regular eigenvalue problem.

Proposition 3.1. *Let $f_{m+1}(\mathbf{x}') = x_i - \lambda$, then there exists a block partitioning of \mathbf{M} in (7) as:*

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix}, \quad (8)$$

such that (7) can be converted to an eigenvalue problem of the form $\mathbf{X} \mathbf{b}' = \lambda \mathbf{b}'$.

Proof: In order to block partition the columns in (8) we need to partition B as $B = B_\lambda \sqcup B_c$ where

$$B_\lambda = B \cap T_{m+1}, \quad B_c = B - B_\lambda. \quad (9)$$

Let us order the monomials in B , such that $\mathbf{b} = \text{vec}(B) = [\text{vec}(B_\lambda) \text{vec}(B_c)]^T = [\mathbf{b}_1 \mathbf{b}_2]^T$. Such a partition of \mathbf{b} induces a column partition of \mathbf{M} (7). We row partition \mathbf{M} such that the lower block is row-indexed by monomial multiples of $f_{m+1}(\mathbf{x}')$ which are linear in λ (i.e. $\mathbf{x}^{\alpha_j}(x_i - \lambda)$, $\mathbf{x}^{\alpha_j} \in T_{m+1}$) while the upper block is indexed by monomial multiples of $f_1(\mathbf{x}'), \dots, f_m(\mathbf{x}')$. Such a row and column partition of \mathbf{M} gives us a block partition as in (8). As $[\mathbf{M}_{11} \mathbf{M}_{12}]$ contains polynomials independent of the λ and $[\mathbf{M}_{21} \mathbf{M}_{22}]$ contains polynomials of the form $\mathbf{x}^{\alpha_j}(x_i - \lambda)$ we obtain

$$\begin{aligned} \mathbf{M}_{11} &= \mathbf{A}_{11}, & \mathbf{M}_{12} &= \mathbf{A}_{12} \\ \mathbf{M}_{21} &= \mathbf{A}_{21} + \lambda \mathbf{B}_{21}, & \mathbf{M}_{22} &= \mathbf{A}_{22} + \lambda \mathbf{B}_{22}, \end{aligned} \quad (10)$$

where $\mathbf{A}_{11}, \mathbf{A}_{12}, \mathbf{A}_{21}$ and \mathbf{A}_{22} are matrices dependent only on the coefficients of input polynomials in (6). We assume here

that A_{12} has *full column rank*. Substituting (10) in (8) gives

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ B_{21} & B_{22} \end{bmatrix} \quad (11)$$

We can order monomials so that $\mathbf{T}_{m+1} = \mathbf{b}_1$. Now chosen partition of M implies that M_{21} is column indexed by \mathbf{b}_1 and row indexed by \mathbf{T}_{m+1} . As $\begin{bmatrix} M_{21} & M_{22} \end{bmatrix}$ has rows of form $\mathbf{x}^{\alpha_j}(x_i - \lambda)$, $\mathbf{x}^{\alpha_j} \in T_{m+1} \implies \mathbf{x}^{\alpha_j} \in B_\lambda$. This gives us, $B_{21} = -I$, where I is an identity matrix of size $|B_\lambda|$ and B_{22} is a zero matrix of size $|B_\lambda| \times |B_c|$. This also means that A_{21} is a square matrix of same size as B_{21} . Thus we have a decomposition as

$$M = M_0 + \lambda M_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ -I & 0 \end{bmatrix}, \quad (12)$$

where M is a $p \times \varepsilon$ matrix. If M is a *tall* matrix, so is A_{12} from which we can eliminate extra rows to obtain a square invertible matrix \hat{A}_{12} while preserving the above mentioned structure, as discussed in Section 3.3. Let $\mathbf{b} = [\mathbf{b}_1 \ \mathbf{b}_2]^T$. Then from (7) and (12) we have

$$\begin{aligned} \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ -I & 0 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix} &= \mathbf{0} \\ \implies \hat{A}_{11}\mathbf{b}_1 + \hat{A}_{12}\mathbf{b}_2 &= \mathbf{0}, \\ A_{21}\mathbf{b}_1 + A_{22}\mathbf{b}_2 - \lambda\mathbf{b}_1 &= \mathbf{0} \end{aligned} \quad (13)$$

Eliminating \mathbf{b}_2 from the above pair of equations we obtain

$$\overbrace{(A_{21} - A_{22}\hat{A}_{12}^{-1}\hat{A}_{11})}^x \mathbf{b}_1 = \lambda\mathbf{b}_1, \quad (14)$$

which is the schur complement of \hat{A}_{12} . If A_{12} does not have full column rank, we change the partitioning of columns of M by changing the partitions, $B_\lambda = \{\mathbf{x}^m \in T_{m+1} \mid x_i \mathbf{x}^m \in B\}$ and $B_c = B - B_\lambda$ by exploiting the form of $f_{m+1}(\mathbf{x}')$. This gives us $A_{21} = I$ and $A_{22} = 0$. It also results in a different A_{12} and a different \hat{A}_{12} after removing extra rows. Hence from (12) we have

$$M = M_0 + \lambda M_1 = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ I & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ B_{21} & B_{22} \end{bmatrix}, \quad (15)$$

which is substituted in (7) to get $\hat{A}_{11}\mathbf{b}_1 + \hat{A}_{12}\mathbf{b}_2 = \mathbf{0}$ and $\lambda(B_{21}\mathbf{b}_1 + B_{22}\mathbf{b}_2) + \mathbf{b}_1 = \mathbf{0}$. Eliminating \mathbf{b}_2 from these equations we get an alternate eigenvalue formulation:

$$(B_{21} - B_{22}\hat{A}_{12}^{-1}\hat{A}_{11})\mathbf{b}_1 = -(1/\lambda)\mathbf{b}_1. \quad \square \quad (16)$$

We note that (14) defines our proposed solver. Here we can extract solutions to x_1, \dots, x_n by computing eigenvectors of X . If in case \hat{A}_{12} is not invertible, we can use the alternate formulation (16) and extract solutions in a similar manner. It is worth noting that the speed of execution of the solver depends on the size of $\mathbf{b}_1 (=|B_\lambda|)$ as well the size

of \hat{A}_{12} while the accuracy of the solver largely depends on the matrix to be inverted *i.e.* \hat{A}_{12} . Hence, in next section we outline a generalized algorithm for computing a set of monomial multiples T as well as the monomial basis B that leads to matrix M satisfying Proposition 3.1.

3.2. Computing a monomial basis

Our approach is based on the algorithm explored in [18] for computing a monomial basis B for a sparse resultant.

We briefly define the basic terms related to convex polytopes used for computing a monomial basis B . A Newton polytope of a polynomial $\text{NP}(f)$ is defined as a convex hull of the exponent vectors of the monomials occurring in the polynomial (also known as the support of the polynomial). Hence, we have $\text{NP}(f_i) = \text{Conv}(A_i)$ where $A_i = \{\alpha \mid \alpha \in \mathbb{Z}_{\geq 0}^n\}$ is the set of all integer vectors that are exponents of monomials with non-zero coefficients in f_i . A Minkowski sum of any two convex polytopes P_1, P_2 is defined as $P_1 + P_2 = \{p_1 + p_2 \mid \forall p_1 \in P_1, p_2 \in P_2\}$. An extensive treatment of polytopes can be found in [10]. The algorithm by Heikkilä [18] basically computes the Minkowski sum of the Newton polytopes of a subset of input polynomials, $Q = \Sigma_i \text{NP}(f_i(\mathbf{x}))$. The set of integer points in the interior of Q defined as $B = \mathbb{Z}_{n-1} \cap (Q + \delta)$, where δ is a small random displacement vector, can provide a monomial basis B satisfying the constraint (2). Our proposed approach computes B as a prospective monomial basis in a similar way, albeit for a modified polynomial system (6). Next we describe our approach and provide a detailed algorithm for the same in the supplementary material.

Given a system of $m(\geq n)$ polynomials (1) in n variables $X = \{x_1, \dots, x_n\}$ we introduce a new variable λ and create n augmented systems $F' = \{f_1, \dots, f_m, x_i - \lambda\}$ for each variable $x_i \in X$. Then we compute the support $A_j = \text{supp}(f_j)$ and the Newton polytope $\text{NP}(f_j) = \text{conv}(A_j)$ for each polynomial $f_j \in F'$. The unit simplex $\text{NP}_0 \subset \mathbb{Z}^n$ is also computed. For each polynomial system F' , we consider each subset of polynomials $F_{\text{sub}} \subset F'$ and compute its Minkowski sum, $Q = \text{NP}_0 + \Sigma_{f \in F_{\text{sub}}} \text{NP}(f)$. Then for various displacement vectors δ we try to compute a candidate monomial basis B as the set of integer points inside $Q + \delta$.

From B we compute $T_j = \{t \in \mathbb{Z}^n \mid t + \text{supp}(f_j) \subset B\}, \forall f_j \in F'$. Assuming T to be the set of monomial multiples for input polynomials, our approach tests that $\Sigma_{j=1}^{m+1} |T_j| \geq |B|$, $\min_j |T_j| > 0$ and $\text{rank}(M) = |B|$. If successful, we compute the coefficient matrix M indexed by B and T as in Section 3.1 and partition B into sets $B_\lambda = B \cap T_{m+1}$ (or $B_\lambda = \{\mathbf{x}^m \in T_{m+1} \mid x_i \mathbf{x}^m \in B\}$ if we need to use the alternate formulation (16)) and $B_c = B - B_\lambda$. If the submatrix of M column indexed by B_c and row indexed by $T_1 \cup \dots \cup T_m$ has full column rank then we add B to the list of favourable monomial bases.

Our algorithm then goes through all of the favorable

monomial bases so computed and selects the smallest monomial basis B among them along with the corresponding set of monomial multiples T from which the coefficient matrix M is constructed as described in Section 3.1.

Next, we list the prominent features of our approach and how they seek to address the shortcomings of [12, 18]:

1. We attempt to generate the smallest basis B by testing adding an extra polynomial (5) of a special form $x_i - \lambda$ for each i in $1, \dots, n$.
2. We explicitly test for rank of M for each candidate basis B to ensure that we have a full rank solver. This addresses the issue of rank-deficient solvers in [18].
3. The partition of monomial basis, $B = B_\lambda \sqcup B_c$ (9) (or the alternate partition of B as described in Proposition 3.1) highlights our approach that leads to a favourable decomposition of the coefficient matrix M as in (12), for solving (7) as an eigenvalue problem. This helps us compute much smaller and more stable solvers as compared to ones generated in [12, 13, 18].
4. The special form of the extra polynomial aids us to construct M that is largely smaller than the one constructed by general u -resultant approach in [12].
5. Our method can generate solvers for $m \geq n$ in (1).

3.3. Removing columns from coefficient matrix

The next step in our method is to attempt to reduce the size of the coefficient matrix M computed in the previous section. For this, we select columns of M one by one in a random order to test for their removal. For each such column, we select rows (say r_1, \dots, r_k) that contain non-zero entries in the column and also consider all columns (say c_1, \dots, c_l) that have non-zero entries in r_1, \dots, r_k . Then we can remove these k rows and l columns from M only if the following conditions hold true for the resulting reduced matrix M_{red} . This also means that we would be removing monomials from B that index c_1, \dots, c_l and removing monomials from T that index r_1, \dots, r_k .

1. After eliminating the monomials from T , we require that there is at least one monomial left in each T_i .
2. If M is of size $p \times \varepsilon$, the reduced matrix M_{red} would be of size $(p - k) \times (\varepsilon - l)$. Then we require $p - k \geq \varepsilon - l$ and $\text{rank}(M_{\text{red}}) = \varepsilon - l$.
3. M_{red} must be block partitioned and decomposed as in Proposition 3.1.

We repeat the above process until there are no more columns that can be removed. We note that the last condition is important as it ensures that at each stage, the reduced matrix can still be partitioned and decomposed into an eigenvalue formulation (14) (or alternately (16)). Now, reusing the notation, let's denote M to be the reduced coefficient matrix and denote B and T to be reduced monomial basis and set of monomial multiples, respectively.

If M still has more rows than columns, we transform it into a square matrix by removing extra rows (say q_1, \dots, q_j) and the monomials from T indexing these rows. These rows are chosen in a way so that the three conditions mentioned above are still satisfied. Moreover, our proposed approach first tries to remove as many rows as possible from the lower block, indexed by T_{m+1} . This is to reduce $|T_{m+1}| (= |B_\lambda|)$ as much as possible and ensure that the matrix A_{21} and hence X (14) (or in (16)) for eigenvalue problem has as small size as possible. Then, if there are more rows still to be removed, the rest are randomly chosen from the upper block indexed by $\{T_1, \dots, T_m\}$. Detailed algorithms for these two steps of matrix reduction are provided in the supplementary material. But we note that at the end of these two steps, we have the sparse resultant matrix, M satisfying (7) which is then reduced to the eigenvalue formulation (14) or to the alternate formulation (16).

4. Experiments

We evaluate the performance of our method by comparing the stabilities as well as computational complexities of the solvers generated using our method with the state-of-art Gröbner basis solvers for many interesting minimal problems. The minimal problems selected for comparison represent a huge variety of relative and absolute pose problems and correspond to that studied in [32]. Results for additional problems are provided in the supplementary material.

4.1. Evaluation

The comparison of the computational complexity of minimal solvers is based on the sizes of matrix templates to be solved. E.g. a solver of size 11×20 in the table means inverting a 11×11 matrix and then a computation of $20 - 11 = 9$ eigenvalues and eigenvectors. So in Table 1 we compare the size of templates in our resultant-based solvers with the templates used in state-of-the-art Gröbner basis solvers as well as in the original solvers proposed by the respective authors (see column 3). The Gröbner basis solvers used for comparison include the solvers generated using the approach in [29], the Gröbner fan and heuristic-based approaches in [32]. As we can see from Table 1, our new resultant-based approach leads to the smallest templates and hence fastest solvers for most of the minimal problems while for only a few problems our generated solver is slightly larger than the state-of-the-art solver based on the Gröbner fan or the heuristic-based method [32]. For some solvers though we have a slightly larger eigenvalue problem, the overall template size is considerably smaller. E.g. in the problem of estimating the relative pose and radial distortion parameter from 6pt correspondences [24] we have an eigenvalue problem of size 56×56 and matrix inversion of size 39×39 whereas the heuristic-based solver has a 52×52 eigenvalue problem but inversion of a larger

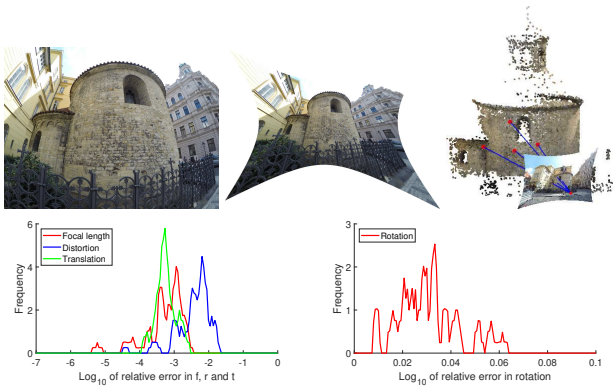


Figure 1. Top row: Example of an input image (left). Undistorted image using the proposed resultant-based P4Pfr solver (middle). Input 3D point cloud and an example of registered camera (right). Bottom row: Histograms of errors for 62 images. The measured errors are (left) the Log_{10} relative focal length $|f - f_{GT}|/f_{GT}$, radial distortion $|k - k_{GT}|/|k_{GT}|$, and the relative translation error $\|\vec{t} - \vec{t}_{GT}\|/\|\vec{t}_{GT}\|$, and (right) the rotation error in degrees.

matrix of size 53×53 . For this problem the resultant-based solver is slightly faster than the state-of-the-art heuristic-based solver [32]. Note that for this problem we failed to generate a Gröbner fan solver [32] in reasonable time. It is worth noting that here we do not compare our solvers’ sizes with resultant-based solvers generated by original versions of [18] and [12]. These methods can not be directly applied to most of the studied minimal problems as they can not handle more equations than unknowns. With [18] we also failed to generate full rank solvers for some problems. Even after proposing extensions to these methods [18, 12], the generated solvers were larger than ours, and GEP involved in [18] led also to many unwanted solutions. We give the sizes of these solvers in supplementary material along with a brief description of our proposed improvements to [18].

We evaluate and compare the stabilities of our solvers from Table 1 with Gröbner basis solvers. As it is not feasible to generate scene setups for all considered problems, we instead evaluate the stability of minimal solvers using 5K instances of random data points. Stability measures include mean and median of Log_{10} of normalized equation residuals for computed solutions as well as the solvers failures as a % of 5K instances for which at least one solution has a normalized residual $> 10^{-3}$. These measures on randomly generated inputs have been shown to be sufficiently good indicators of solver stabilities [29]. Table 2 shows stabilities of solvers for six minimal problems selected from Table 1. We note that for the “Rel.pose $\lambda+E+\lambda$ ” problem, our solver is not only faster, but also more stable than the state-of-the-art solvers whose histogram is provided in the supplementary material along with the stabilities for the remaining problems and the histograms of their residuals. In general, our new method generates solvers that are stable with only very few failures.

Note that as our new solvers are solving the same formulations of problems as the existing state-of-the-art solvers, the performance on noisy measurements and real data would be the same as the performance of the state-of-the-art solvers. The only difference in the performance comes from numerical instabilities that already appear in the noise-less case and are detailed in Table 2 (fail%). For performance of the solvers in real applications we refer the reader to papers where the original formulations of the studied problems were presented (see Table 1, column 3). Here we select two interesting problems, *i.e.* one relative and one absolute pose problem, and perform experiments on synthetically generated scenes and on real images, respectively.

E+ f λ solver on synthetic scenes: We study the numerical stability of the new resultant-based solver for the problem of estimating the relative pose of one calibrated and one camera with unknown focal length and radial distortion from 7-point correspondences, *i.e.* the Rel. pose E+ f λ 7pt problem from Table 1. We considered the formulation “elim. λ ” proposed in [32] that leads to the smallest solvers. We studied the performance on noise-free data and compared it to the results of Gröbner basis solvers from Table 1.

We generated 10K scenes with 3D points drawn uniformly from a $[-10, 10]^3$ cube. Each 3D point was projected by two cameras with random feasible orientation and position. The focal length of the first camera was randomly drawn from the interval $f_{gt} \in [0.5, 2.5]$ and the focal length of the second camera was set to 1, *i.e.*, the second camera was calibrated. The image points in the first camera were corrupted by radial distortion following the one-parameter division model. The radial distortion parameter λ_{gt} was drawn at random from the interval $[-0.7, 0]$ representing distortions of cameras with a small distortion up to slightly more than GoPro-style cameras. A graph for the Log_{10} of the relative errors of the distortion parameter λ as well as the focal length f are provided in the supplementary material.

P4Pfr solver on real images: We evaluated the resultant-based solver for a practical problem of estimating the absolute pose of camera with unknown focal length and radial distortion from four 2D-to-3D point correspondences, *i.e.* the P4Pfr solver, on real data. We consider the *Rotunda* dataset, which was proposed in [26] and in [31] it was used for evaluating P4Pfr solvers. This dataset consists of 62 images captured by a GoPro Hero4 camera. Example of an input image from this dataset (left) as well as undistorted (middle) and registered image (right) using our new solver, is shown in Figure 1 (top). The Reality Capture software [1] was used to build a 3D reconstructions of this scene. We used the 3D model to estimate the pose of each image using the new P4Pfr resultant-based solver (28×40) in a RANSAC framework. Similar to [31], we used the camera and distortion parameters obtained from [1] as ground

Problem	Our	Original*	[29]	GFan [32]	(#GB)	Heuristic [32]
Rel. pose F+ λ 8pt ^(‡) (8 sols.)	7 × 16	12 × 24 [22]	11 × 19	11 × 19	(10)	7 × 15
Rel. pose E+f 6pt (9 sols.)	11 × 20	21 × 30 [3]	21 × 30	11 × 20	(66)	11 × 20
Rel. pose f+E+f 6pt (15 sols.)	12 × 30	31 × 46 [24]	31 × 46	31 × 46	(218)	21 × 36
Rel. pose E+ λ 6pt (26 sols.)	14 × 40	48 × 70 [22]	34 × 60	34 × 60	(846)	14 × 40
Stitching f λ +R+f λ 3pt (18 sols.)	18 × 36	54 × 77 [34]	48 × 66	48 × 66	(26)	18 × 36
Abs. Pose P4Pfr (16 sols.)	52 × 68	136 × 152 [4]	140 × 156	54 × 70	(1745)	54 × 70
Abs. Pose P4Pfr (elim. f) (12 sols.)	28 × 40	28 × 40 [31]	48 × 60	28 × 40	(699)	28 × 40
Rel. pose λ +E+ λ 6pt ^(‡) (52 sols.)	39 × 95	238 × 290 [24]	149 × 201	-	?	53 × 105
Rel. pose λ_1 +F+ λ_2 9pt (24 sols.)	90 × 117	179 × 203 [24]	189 × 213	87 × 111	(6896)	87 × 111
Rel. pose E+f λ 7pt (19 sols.)	61 × 80	200 × 231[22]	181 × 200	69 × 88	(3190)	69 × 88
Rel. pose E+f λ 7pt (elim. λ) (19 sols.)	22 × 41	-	52 × 71	37 × 56	(332)	24 × 43
Rel. pose E+f λ 7pt (elim. f λ) (19 sols.)	51 × 70	51 × 70 [27]	51 × 70	51 × 70	(3416)	51 × 70
Abs. pose quivers ^(†) (20 sols.)	68 × 92	372 × 386 [21]	203 × 223	-	?	68 × 88
Rel. pose E angle+4pt (20 sols.)	-	270 × 290 [33]	266 × 286	-	?	183 × 203
Abs. pose refractive P5P ^(†) (16 sols.)	68 × 93	280 × 399 [15]	199 × 215	112 × 128	(8659)	199 × 215
Unsynch. Rel. pose (16 sols.)	150 × 168	633 × 649[2]	467 × 483	-	?	299 × 315

Table 1. Comparison of solver sizes for some minimal problems. Missing entries are when we failed to generate a solver. (*): Sizes for the original formulations. (†): Input polynomials were eliminated using G-J elimination before generating a solver using our resultant method as well as solvers based on [29], the Gröbner fan-based solver [32] and the heuristic-based solver [32]. (‡): Solved using the alternate eigenvalue formulation (16).

Problem	Our			[29]			Heuristic [32]		
	mean	med.	fail(%)	mean	med.	fail(%)	mean	med.	fail(%)
Rel. pose f+E+f 6pt	-12.55	-12.90	0.52	-12.09	-12.53	2.36	-12.05	-12.48	1.44
Abs. Pose P4Pfr (elim. f)	-12.86	-13.08	0	-12.59	-12.85	0	-12.73	-13.00	0.02
Rel. pose λ +E+ λ 6pt	-8.99	-9.33	14.66	-6.92	-7.45	25.9	-8.13	-8.73	26.46
Rel. pose E+f λ 7pt ^(‡)	-11.29	-11.59	0.36	-10.69	-11.13	7.58	-	-	-
Rel. pose E+f λ 7pt (elim. λ)	-12.53	-12.95	2.34	-11.99	-12.35	0.44	-11.05	-11.84	5.70
Abs. pose refractive P5P ^(†)	-13.03	-13.25	0	-12.45	-12.79	0.10	-12.23	-12.53	0.08

Table 2. Stability comparison for solvers generated by our new method, solvers generated using [29] and heuristic-based solvers [32] on some interesting minimal problems. Mean and median are computed from Log_{10} of normalized equation residuals. (†): Solvers generated after Gauss-Jordan(G-J) elimination of input polynomials. (‡): Failed to extract solutions to all variables for the heuristic-based solver [32].

truth for the experiment. Figure 1 (bottom) shows the errors for the focal length, radial distortion, and the camera pose. Overall the errors are quite small, *e.g.* most of the focal lengths are within 0.1% of the ground truth and almost all rotation errors are less than 0.1 degrees, which shows that our new solver works well for real data. These results (summarized in the supplementary material) are consistent with the results of the P4Pfr solver presented in [31], which was tested on the same dataset. The slightly different results reported in [31] are due to RANSAC’s random nature and a slightly different P4Pfr formulation (40x50) used in [31].

5. Conclusion

Here we propose a novel algorithm for generating efficient minimal solvers based on sparse resultants that achieves significant improvements over existing resultant-based methods in terms of efficiency of the generated solvers. Our experiments on many minimal problems on real and synthetic

scenes show that the new method is a competitive alternative to the highly optimised Gröbner basis methods. The fact that new resultant-based solvers have for many problems the same size as the state-of-the-art heuristic or GFan solvers, shows that these solvers are maybe already “optimal” w.r.t. template sizes. On the other hand, there is no one general method (GFan/heuristic/resultant), which provably returns the smallest solver for every problem and we believe that especially for complex problems all methods have to be tested when trying to generate the “best” solver.

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