Supplementary material for: Globally Optimal Contrast Maximisation for Event-based Motion Estimation

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A. Geometric derivations of the elliptical region

Here we present the analytic form of the centre $\mathbf{c}$, semi-major axis $\mathbf{y}$, and semi-minor axis $\mathbf{z}$ of the elliptical region $\mathcal{L}$ (see Sec. 3.1 in the main text) following the method in [1, 2] (subscript i and explicit dependency on $\mathbb{B}$ are omitted for simplicity). See Fig. 2a in the main text for a visual representation of the aforementioned geometric entities.

1. Calculate direction of the cone-beam

$$\hat{\mathbf{u}} = \frac{\mathbf{R}(t;\boldsymbol{\omega}_\mathbf{c})\tilde{\mathbf{u}}}{\|\mathbf{R}(t;\boldsymbol{\omega}_\mathbf{c})\tilde{\mathbf{u}}\|_2}, \tag{1}$$

its radius

$$r = \sin\alpha(\mathbb{B}), \tag{2}$$

and the norm vector to the image plane $\hat{\mathbf{n}} = [0\ 0\ 1]^T$.

2. Calculate the semi-major axis direction within the cone-beam

$$\hat{\mathbf{y}} = \frac{\hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \hat{\mathbf{n}})}{\|\hat{\mathbf{u}} \times (\hat{\mathbf{u}} \times \hat{\mathbf{n}})\|} \tag{3}$$

and semi-minor axis direction

$$\hat{\mathbf{z}} = \frac{\hat{\mathbf{y}} \times \hat{\mathbf{n}}}{\|\hat{\mathbf{y}} \times \hat{\mathbf{n}}\|}. \tag{4}$$

3. Calculate the intersecting points between the ray with the direction of the semi-major axis and the cone-beam

$$\begin{aligned}\mathbf{y}^{(a)} &= \hat{\mathbf{u}} - r\hat{\mathbf{y}} \\ \mathbf{y}^{(b)} &= \hat{\mathbf{u}} + r\hat{\mathbf{y}},\end{aligned} \tag{5}$$

and the analogous points for the semi-minor axis

$$\begin{aligned}\mathbf{z}^{(a)} &= \hat{\mathbf{u}} - r\hat{\mathbf{z}} \\ \mathbf{z}^{(b)} &= \hat{\mathbf{u}} + r\hat{\mathbf{z}}.\end{aligned} \tag{6}$$

4. Obtain $\mathbf{y}'^{(a)}$, $\mathbf{y}'^{(b)}$, $\mathbf{z}'^{(a)}$ and $\mathbf{z}'^{(b)}$ as the projection of (5) and (6) into the image plane with the intrinsic matrix $\mathbf{K}$.

5. Calculate $\mathbf{c} = 0.5(\mathbf{y}'^{(a)} + \mathbf{y}'^{(b)})$, $\mathbf{y} = \|\mathbf{y}'^{(a)} - \mathbf{y}'^{(b)}\|_2$, and $\mathbf{z} = \|\mathbf{z}'^{(a)} - \mathbf{z}'^{(b)}\|_2$.

B. Proofs

We state our integer quadratic problem again.

$$\begin{aligned}\overline{S}_d^*(\mathbb{B}) = \max_{\mathbf{Z}\in\{0,1\}^{N\times K}} \quad & \sum_{k=1}^{K}\left(\sum_{i=1}^{N}\mathbf{Z}_{i,k}\mathbf{M}_{i,k}\right)^2 \\ \text{s.t.} \quad & \mathbf{Z}_{i,k} \le \mathbf{M}_{i,k}, \ \ \forall i,k, \\ & \sum_{k=1}^{K}\mathbf{Z}_{i,k} = 1, \ \ \forall i.\end{aligned} \tag{IQP}$$

B.1. Proof of Lemma 1 in the main text

Lemma 1.

$$\overline{H}_c(\mathbf{x}_j;\mathbb{B}) \ge \max_{\boldsymbol{\omega}\in\mathbb{B}} H_c(\mathbf{x}_j;\boldsymbol{\omega}) \tag{7}$$

with equality achieved if $\mathbb{B}$ is singleton, i.e., $\mathbb{B} = \{\boldsymbol{\omega}\}$.

Proof. This lemma can be demonstrated by contradiction. Let $\boldsymbol{\omega}^*$ be the optimiser for the RHS of (7). If

$$H_c(\mathbf{x}_j;\boldsymbol{\omega}^*) > \overline{H}_c(\mathbf{x}_j;\mathbb{B}), \tag{8}$$

it follows from the definition of pixel intensity (Eq. (1) in the main text) and its upper bound (Eq. (23) in the main text) that

$$\|\mathbf{x}_j - f(\mathbf{u}_i, t_i, \boldsymbol{\omega}^*)\| < \max\left(\|\mathbf{x}_j - \mathbf{c}_i(\mathbb{B})\| - \|\mathbf{y}_i(\mathbb{B})\|, 0\right), \tag{9}$$

for at least one $i = 1,\dots,N$.

In words, the shortest distance between $\mathbf{x}_j$ and the disc $\mathcal{D}_i(\mathbb{B})$ is greater than the distance between $\mathbf{x}_j$ and the optimal position $f(\mathbf{u}_i, t_i, \boldsymbol{\omega}^*)$. However, $f(\mathbf{u}_i, t_i, \boldsymbol{\omega}^*)$ is always inside the disc $\mathcal{D}_i(\mathbb{B})$, and hence Eq. (9) cannot hold. If $\mathbb{B} = \{\boldsymbol{\omega}\}$, then from definition (23) in the main text $\overline{H}_c(\mathbf{x}_j;\mathbb{B}) = H_c(\mathbf{x}_j;\boldsymbol{\omega})$. □

B.2. Proof of Lemma 2 in the main text

Lemma 2.

$$\overline{S}_d^*(\mathbb{B}) \ge \max_{\boldsymbol{\omega}\in\mathbb{B}} \sum_{j=1}^{P} H_d(\mathbf{x}_j;\boldsymbol{\omega})^2, \tag{10}$$

with equality achieved if $\mathbb{B}$ *is singleton, i.e.,* $\mathbb{B} = \{\boldsymbol{\omega}\}$.

Proof. We pixel-wisely reformulate IQP:

$$\overline{S}_d^*(\mathbb{B}) = \max_{\mathbf{Q}\in\{0,1\}^{N\times P}} \sum_{j=1}^{P}\left(\sum_{i=1}^{N}\mathbf{Q}_{i,j}\right)^2 \quad \text{s.t.} \quad \mathbf{Q}_{i,j} \le \mathbf{T}_{i,j}, \ \forall i,j, \quad \sum_{j=1}^{P}\mathbf{Q}_{i,j} = 1, \ \forall i, \tag{P-IQP}$$

and we express the RHS of (10) as a mixed integer quadratic program:

$$\max_{\boldsymbol{\omega}\in\mathbb{B},\mathbf{Q}\in\{0,1\}^{N\times P}} \sum_{j=1}^{P}\left(\sum_{i=1}^{N}\mathbf{Q}_{i,j}\right)^2 \quad \text{s.t.} \quad \mathbf{Q}_{i,j} = \mathbb{I}(f(\mathbf{u}_i,t_i;\boldsymbol{\omega}) \text{ in } \mathbf{x}_j), \ \forall i,j. \tag{MIQP}$$

Problem P-IQP is a relaxed version of MIQP - hence (10) holds - as for every $\boldsymbol{e}_i$, the feasible pixel $\mathbf{x}_j$ is in $\mathcal{D}_i(\mathbb{B})$; whereas for MIQP, the feasible pixel is dictated by a single $\boldsymbol{\omega} \in \mathbb{B}$. If $\mathbb{B}$ collapses into $\boldsymbol{\omega}$, every event $\boldsymbol{e}_i$ can intersect only one pixel $\mathbf{x}_j$, hence $\mathbf{T}_{i,j} = \mathbb{I}(f(\mathbf{u}_i,t_i;\boldsymbol{\omega}) \text{ in } \mathbf{x}_j), \ \forall i,j$; $\sum_{j=1}^{P}\mathbf{T}_{i,j} = 1, \ \forall i$; and $\sum_{j=1}^{P}\mathbf{Q}_{i,j} = 1 \implies \mathbf{Q}_{i,j} = \mathbf{T}_{i.j}, \forall i$; therefore, MIQP is equivalent to P-IQP if $\mathbb{B} = \{\boldsymbol{\omega}\}$.

□

B.3. Proof of Lemma 3 in the main text

Lemma 3. *Problem IQP has the same solution if* $\mathbf{M}$ *is replaced with* $\mathbf{M}'$.

Proof. We show that removing an arbitrary non-dominant column from $\mathbf{M}$ does not change the solution of IQP. Without loss of generality, assume the last column of $\mathbf{M}$ is non-dominant. Equivalent to solving IQP on $\mathbf{M}$ without its last column is the following IQP reformulation:

$$\overline{S}_d^*(\mathbb{B}) = \max_{\mathbf{Z}\in\{0,1\}^{N\times K}} \sum_{k=1}^{K-1}\left(\sum_{i=1}^{N}\mathbf{Z}_{i,k}\mathbf{M}_{i,k}\right)^2 + \tag{11a}$$

$$\left(\sum_{i=1}^{N}\mathbf{Z}_{i,K}\mathbf{M}_{i,K}\right)^2 \tag{11b}$$

$$\text{s.t.} \quad \mathbf{Z}_{i,k} \le \mathbf{M}_{i,k}, \ \forall i,k, \tag{11c}$$

$$\sum_{k=1}^{K}\mathbf{Z}_{i,k} = 1, \ \forall i, \tag{11d}$$

$$\mathbf{Z}_{i,K} = 0, \ \forall i, \tag{11e}$$

which is same as IQP but with additional constraint (11e). Since $\mathbf{M}_{:,K}$ is non-dominant, it must exists a dominant column $\mathbf{M}_{:,\eta}$ such that

$$\mathbf{M}_{i,K} \le \mathbf{M}_{i,\eta}, \ \forall i. \tag{12}$$

Hence, if $\mathbf{M}_{i,K} = 1$, then $\mathbf{M}_{i,\eta} = 1$ must holds $\forall i$. Let $\mathbf{Z}^*$ be the optimiser of IQP with $\mathbf{Z}^*_{i_a,K},\dots,\mathbf{Z}^*_{i_b,K} = 1$. Let define $\mathbf{Z}'^*$ same as $\mathbf{Z}^*$ but with $\mathbf{Z}'^*_{:,K} = \mathbf{0}$ and $\mathbf{Z}'^*_{i_a,\eta},\dots,\mathbf{Z}'^*_{i_b,\eta} = 1$. In words, we "move" the 1 values from the last column to its dominant one. We show that $\mathbf{Z}'^*$ is an equivalent solution (same objective value than $\mathbf{Z}^*$). $\mathbf{Z}'^*$ is feasible since (12) ensures condition (11c), (11d) is not affected by "moving ones" in the same row, and (11e) is true for the definition of $\mathbf{Z}'^*$. Finally we show that

$$\sum_{i=1}^{N}\mathbf{Z}^*_{i,K}\mathbf{M}_{i,K} = \sum_{i=1}^{N}\mathbf{Z}'^*_{i,\eta}\mathbf{M}_{i,\eta} \tag{13}$$

therefore $\mathbf{Z}'^*$ produces same objective value than IQP. We prove (13) by contradiction. Assume exists at least one $i' \notin \{i_a,\dots,i_b\}$ such that $\mathbf{Z}^*_{i',\eta} = 1 \implies \mathbf{Z}'^*_{i',\eta} = 1$. Then, $\mathbf{Z}'^*$ produces a larger objective value than $\mathbf{Z}^*$ which is a contradiction since problem (11) is most restricted than IQP. Thus, removing any arbitrary non-dominant column will not change the solution which implies this is also true if we remove all non-dominant columns (*i.e.*, if we replace $\mathbf{M}$ with $\mathbf{M}'$).

□

B.4. Proof of Lemma 4 in the main text

Lemma 4.

$$\overline{S}_d(\mathbb{B}) \ge \overline{S}_d^*(\mathbb{B}) \tag{14}$$

with equality achieved if $\mathbb{B}$ *is singleton, i.e.,* $\mathbb{B} = \{\boldsymbol{\omega}\}$.

Proof. To prove (14), it is enough to show

$$\overline{S}_d(\mathbb{B}) = \max_{\mathbf{Z}\in\{0,1\}^{N\times K'}} \sum_{k=1}^{K'}\left(\sum_{i=1}^{N}\mathbf{Z}_{i,k}\mathbf{M}'_{i,k}\right)^2 \quad \text{s.t.} \quad \mathbf{Z}_{i,k} \le \mathbf{M}'_{i,k}, \ \forall i,k, \quad \sum_{k=1}^{K'}\sum_{i=1}^{N}\mathbf{Z}_{i,k} = N, \tag{R-IQP}$$

is a valid relaxation of IQP. This is true as the constraint $\sum_{k=1}^{K'}\sum_{i=1}^{N}\mathbf{Z}_{i,k} = N$ in R-IQP is a necessary but not sufficient condition for the constraints $\sum_{k=1}^{K'}\mathbf{Z}_{i,k} = 1, \forall i$ in IQP. If $\mathbb{B}$ collapse into $\boldsymbol{\omega}$, every event $\boldsymbol{e}_i$ can intersect only one CC $\mathcal{G}_k \implies \sum_{k=1}^{K'}\mathbf{Z}_{i,k} = 1$; hence, R-IQP is equivalent to IQP. □

B.5. Proof of lower bound (39) in the main text

Lemma 5.

$$\underline{H}_c(\mathbf{x}_j;\mathbb{B}) \leq \min_{\boldsymbol{\omega}\in\mathbb{B}} H_c(\mathbf{x}_j;\boldsymbol{\omega}) \tag{15}$$

with equality achieved if $\mathbb{B}$ *is singleton, i.e.,* $\mathbb{B} = \{\boldsymbol{\omega}\}$.

Proof. Analogous to Lemma 1, we prove this Lemma by contradiction. Let $\boldsymbol{\omega}^*$ be the optimiser for the RHS of (15). If

$$H_c(\mathbf{x}_j;\boldsymbol{\omega}^*) < \underline{H}_c(\mathbf{x}_j;\mathbb{B}), \tag{16}$$

it follows from the definition of pixel intensity (Eq. (1) in the main text) and its lower bound (Eq. (39) in the main text) that

$$\|\mathbf{x}_j - f(\mathbf{u}_i, t_i, \boldsymbol{\omega}^*)\| > \|\mathbf{x}_j - \mathbf{c}_i(\mathbb{B})\| + \|\mathbf{y}_i(\mathbb{B})\|, \tag{17}$$

for at least one $i = 1, \ldots, N$.

In words, the longest distance between $\mathbf{x}_j$ and the disc $\mathcal{D}_i(\mathbb{B})$ is less than the distance between $\mathbf{x}_j$ and the optimal position $f(\mathbf{u}_i, t_i, \boldsymbol{\omega}^*)$. However, $f(\mathbf{u}_i, t_i, \boldsymbol{\omega}^*)$ is always inside the disc $\mathcal{D}_i(\mathbb{B})$, and hence Eq. (17) cannot hold. If $\mathbb{B} = \{\boldsymbol{\omega}\}$, then from definition (39) in the main text $\underline{H}_c(\mathbf{x}_j;\mathbb{B}) = H_c(\mathbf{x}_j;\boldsymbol{\omega})$.

□

B.6. Proof of lower bound (41) in the main text

Lemma 6.

$$\underline{\mu}_d(\mathbb{B}) \leq \min_{\boldsymbol{\omega}\in\mathbb{B}} \frac{1}{P}\sum_{j=1}^{P} H_d(\mathbf{x}_j;\boldsymbol{\omega}), \tag{18}$$

with equality achieved if $\mathbb{B}$ *is singleton, i.e.,* $\mathbb{B} = \{\boldsymbol{\omega}\}$.

Proof. This lemma can be demonstrated by contradiction. Let $\boldsymbol{\omega}^*$ be the optimiser of the RHS of (18). If

$$\frac{1}{P}\sum_{j=1}^{P} H_d(\mathbf{x}_j;\boldsymbol{\omega}^*) < \underline{\mu}_d(\mathbb{B}), \tag{19}$$

after replacing the pixel intensity and the lower bound pixel value with they definitions (Eqs. (3) and (41) in the main text) in (19), it leads to

$$\sum_{i=1}^{N}\sum_{j=1}^{P} \mathbb{I}(f(\mathbf{u}_i, t_i;\, \boldsymbol{\omega}^*) \text{ lies in pixel } \mathbf{x}_j) \tag{20a}$$

$$< \quad \sum_{i=1}^{N} \mathbb{I}(\mathcal{D}_i \text{ fully lie in the image plane}). \tag{20b}$$

In words, for every warped event $f(\mathbf{u}_i, t_i;\, \boldsymbol{\omega}^*) \in \mathcal{D}_i$ that lies in any pixel $\mathbf{x}_j \in X$ of the image plane, the discs $\mathcal{D}_i$ must fully lie in the image plane. Since (20a) is a less restricted problem than (20b), (19) cannot hold. If $\mathbb{B} = \{\boldsymbol{\omega}\}$, $\mathcal{D}_i = f(\mathbf{u}_i, t_i;\, \boldsymbol{\omega})$; therefore, the two sides in (18) are equivalent.

□

C. Additional qualitative results

Figs. 1, 2 and 3 show additional motion compensation results (Sec. 4.2 in the main text) for subsequences from *boxes*, *dynamic* and *poster*.

References

[1] Rolf Clackdoyle and Catherine Mennessier. Centers and centroids of the cone-beam projection of a ball. *Physics in Medicine & Biology*, 56(23):7371, 2011. 1

[2] Yinlong Liu, Yuan Dong, Zhijian Song, and Manning Wang. 2d-3d point set registration based on global rotation search. *IEEE Transactions on Image Processing*, 28(5):2599–2613, 2018. 1

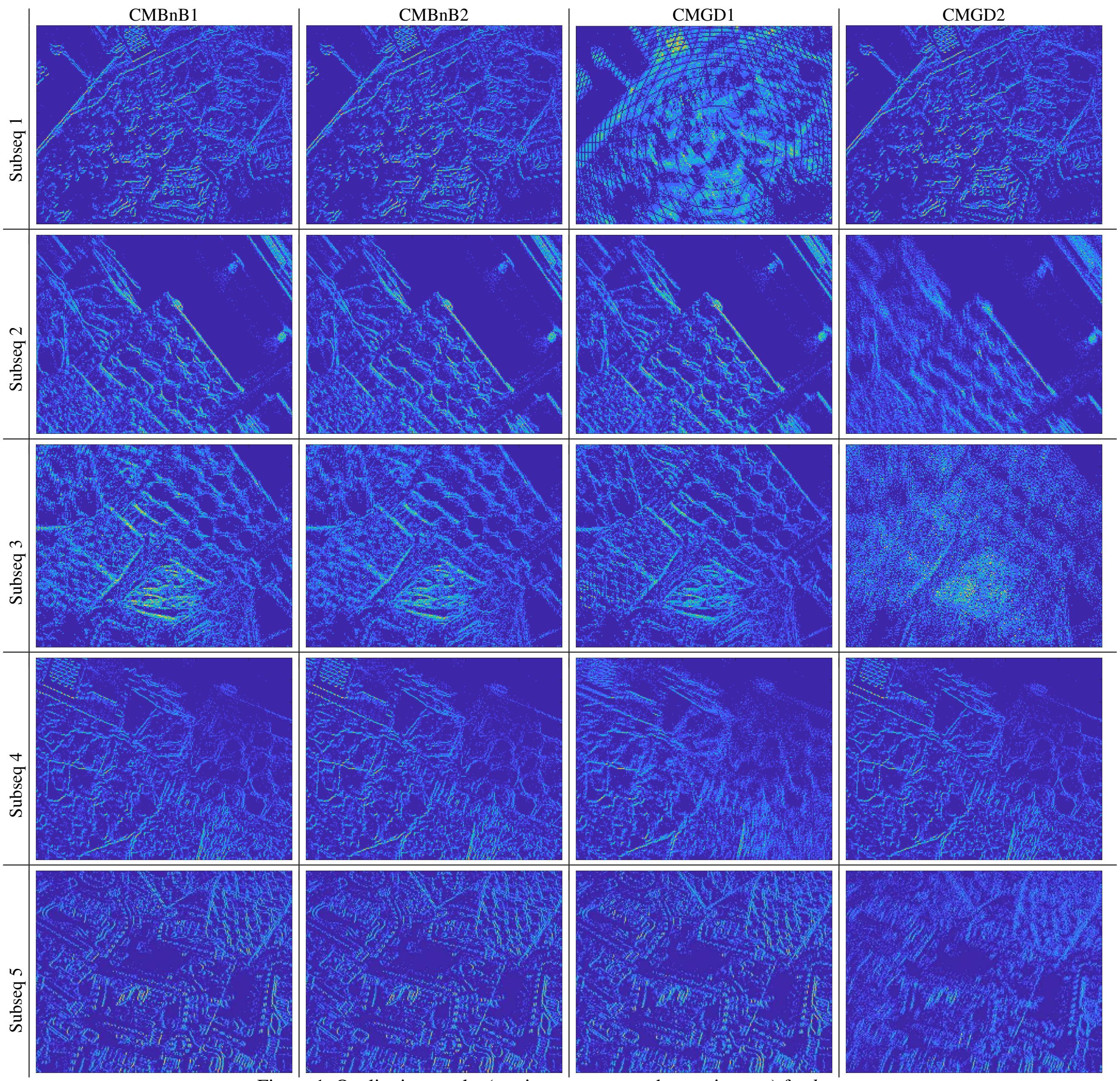

Figure 1. Qualitative results (motion compensated event images) for *boxes*.

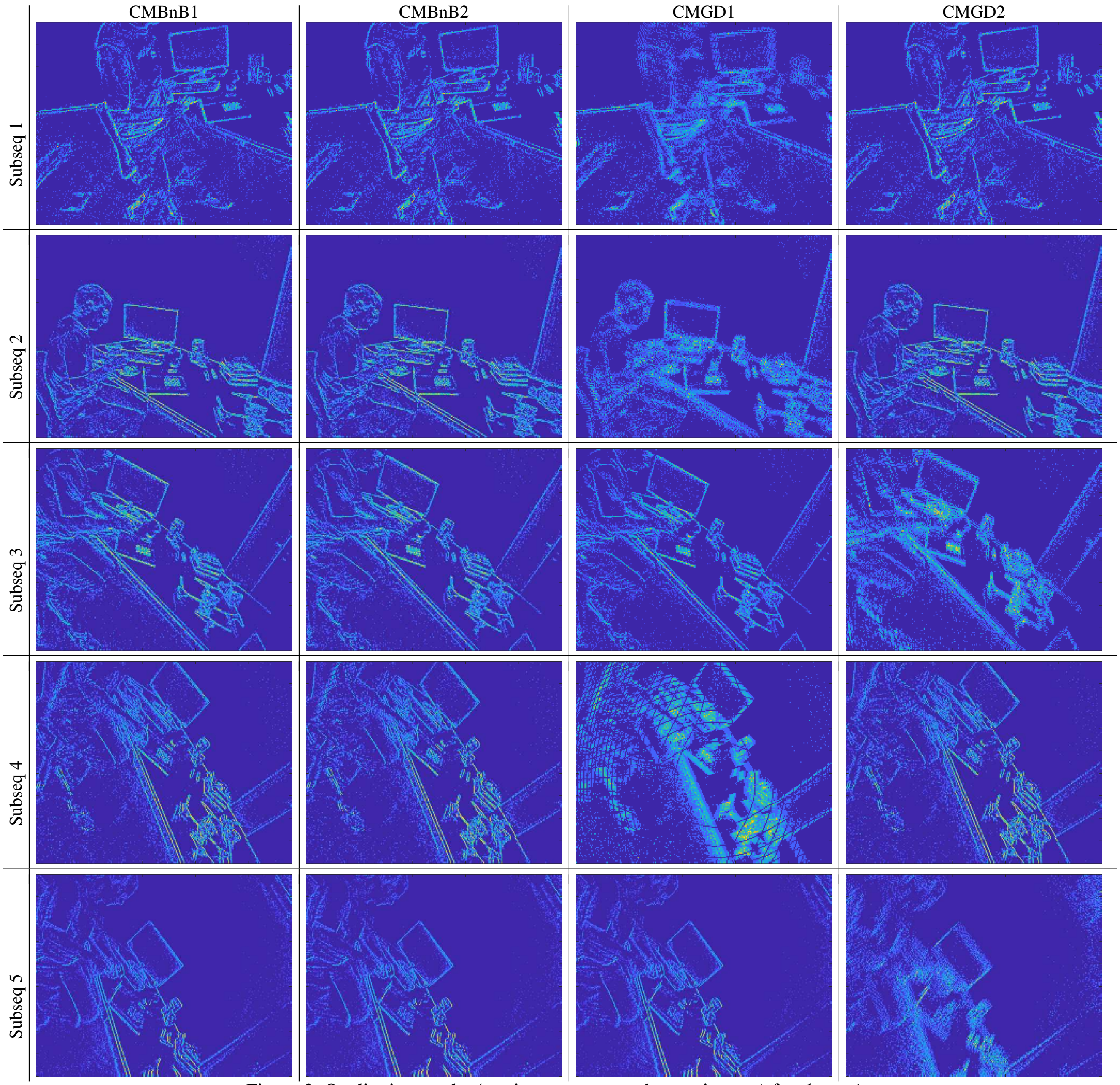

Figure 2. Qualitative results (motion compensated event images) for *dynamic*.

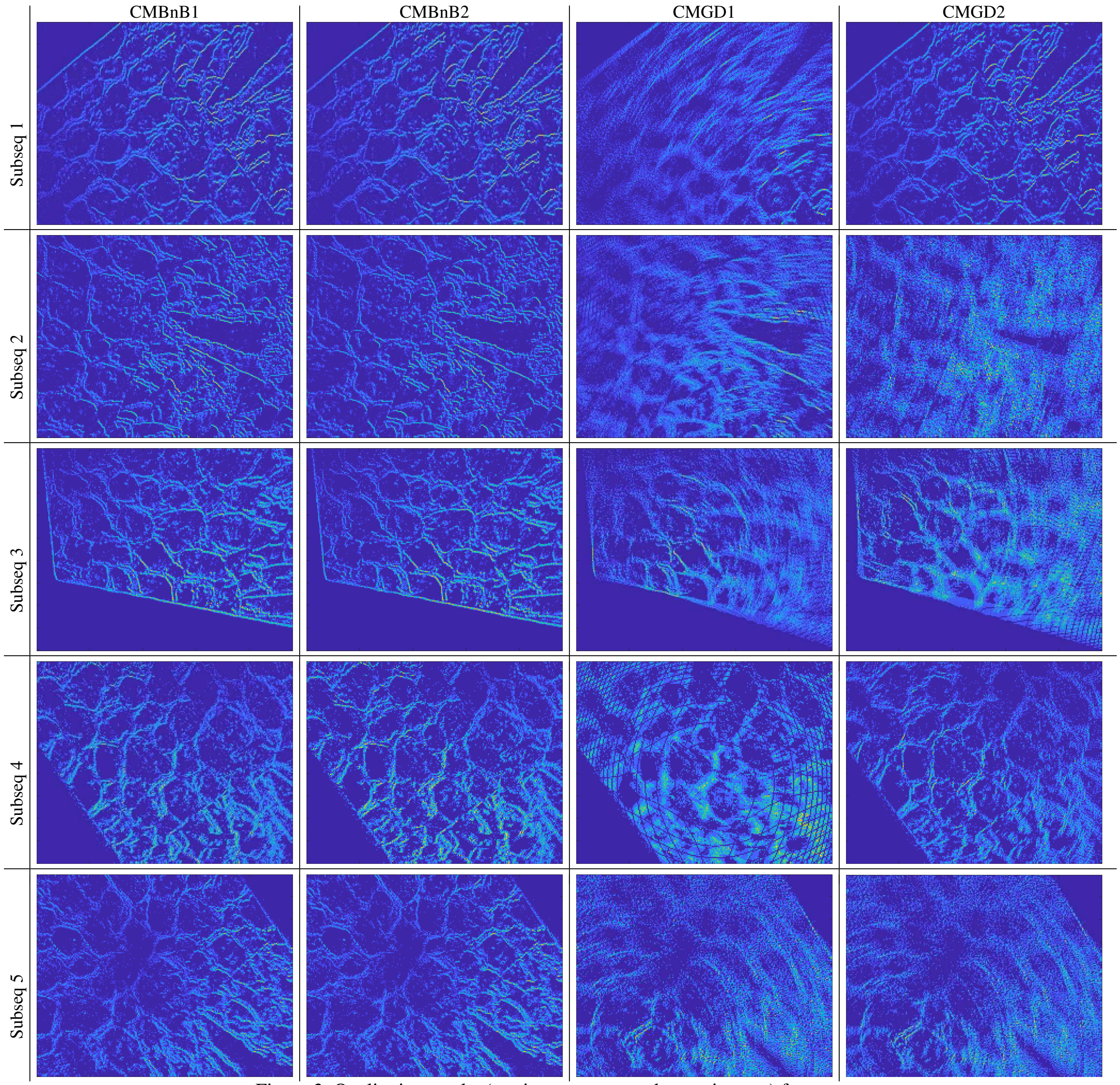

Figure 3. Qualitative results (motion compensated event images) for *poster*.