Pathological Retinal Region Segmentation From OCT Images Using Geometric Relation Based Augmentation

1. Supplementary Material

We assume that the segmentations s'_x of image x are generated from L levels of latent variables z_l. Denoting the generated mask as s for simplicity, we obtain conditional distribution $p(s|x)$ for L latent levels as:

$$p(s|x) = \int p(s|z_1, \cdots, z_L)p(z_1|z_2, x) \cdots \\ p(z_{L-1}|z_L, x)p(z_L|x)dz_1, \cdots, dz_L. \quad (1)$$

The posterior distribution of $p(z|s, x)$ is approximated using a variational approximation $q(z|s, x)$ where $z = \{z_1, ..., z_L\}$. $\log p(s|x) = L(s|x) + KL(q(z|s, x)||p(z|s, x))$, where L is the evidence lower bound, and $KL(\cdot, \cdot)$ is the Kullback-Leibler divergence.

Since $KL(\cdot, \cdot) \geq 0$, L is a lower bound on the conditional log probability when the approximation q matches the posterior exactly. Using the decomposition in Eq.1,

$$\mathcal{L} = \mathbb{E}_{q(z_1,...,z_L|x,s)}\left[\log p(s|z_1, ..., z_L)\right] - \\ \alpha_L KL\left[q(z_L|s, x)||p(z_L|x)\right] - \\ \sum_{l=1}^{L-1} \alpha_l \underset{q(z_{l+1}|s,x)}{\mathbb{E}}\left[KL\left[q(z_l|z_{l+1}, s, x)||p(z_l|z_{l+1}, x)\right]\right] \quad (2)$$

with α_l = 1. Following standard practice we parameterize the prior and posterior distributions as axis aligned normal distributions $\mathcal{N}(z|\mu, \sigma)$. Specifically, we define

$$p(z_l|z_{l+1}, x) = \mathcal{N}\left(\mathbf{z}|\phi_l^{(\mu)}\left(z_l|z_{l+1}, x\right), \phi_l^{(\sigma)}\left(z_l|z_{l+1}, x\right)\right) \quad (3)$$

$$q(z_l|z_{l+1}, x, s) = \mathcal{N}\left(\mathbf{z}|\theta_l^{(\mu)}\left(z_l|z_{l+1}, s, x\right), \theta_l^{(\sigma)}\left(z_l|z_{l+1}, s, x\right)\right) \quad (4)$$

1.1. Derivation of Evidence Lower Bound

The complete derivation for Equation 2 is given below: Defining $z = z_1, \cdots, z_L$ we write :

$$\log p(s|x) = \mathcal{L}(s|x) + KL(q(z|s, x)||p(z|s, x)) \quad (5)$$

with

$$\mathcal{L}(s|x) = \mathbb{E}_{q(z|s,x)}\left[p(s|z, x)\right] - KL\left[q(z|s, x)||p(z|s, x)\right] \quad (6)$$

Since $KL\left[q(z|s, x)||p(z|s, x)\right]$ is always positive $\mathcal{L}(s|x)$ is a lower bound on $\log p(s|x)$ with equality when $q(z|s, x) = p(z|s, x)$.

The KL-divergence term in Eq. 6 can be written as

$$KL\left[q(z_1, \cdots, z_L|s, x)||p(z_1, \cdots, z_L|s, x)\right] \\ = KL\left[q(z_L|s, x)q(z_1, \cdots, z_{L-1}|z_L, s, x)||p(z_L|x)p(z_1, \cdots, z_{L-1}|z_L, x)\right] \\ = KL\left[q(z_L|s, x)||p(z_L|x)\right] + \\ \int \cdots \int q(z_1, \cdots, z_L|s, x) \log \frac{q(z_1, \cdots, z_{L-1}|z_L, s, x)}{p(z_1, \cdots, z_{L-1}|z_L, x)} dz_1, \cdots, dz_L, \quad (7)$$

Decomposing the right most term, we finally get

$$\int_{Z_L} \cdots \int_{Z_1} q(z_1, \cdots, z_L|s, x) \log \frac{q(z_1, \cdots, z_{L-1}|z_L, s, x)}{p(z_1, \cdots, z_{L-1}|z_L, x)} dz_1, \cdots, dz_L, \\ = \int_{Z_L} \cdots \int_{Z_1} q(z_1, \cdots, z_L|s, x) \log \frac{\prod_{l=1}^{L-1} q(z_l|z_{l+1}, s, x)}{\prod_{l=1}^{L-1} p(z_l|z_{l+1}, x)} dz_1, \cdots, dz_L, \\ = \int_{Z_L} \cdots \int_{Z_1} q(z_1, \cdots, z_L|s, x) \left[\sum_{l=1}^{L-1} \log \frac{q(z_l|z_{l+1}, s, x)}{p(z_l|z_{l+1}, x)}\right] dz_1, \cdots, dz_L, \\ = \sum_{l=1}^{L-1} \int_{Z_{l+1}} q(z_{l+1}|s, x) \int_{Z_l} q(z_l|z_{l+1}) \log \frac{q(z_l|z_{l+1}, s, x)}{p(z_l|z_{l+1}, x)} dz_l, dz_{l+1}, \\ = \sum_{l=1}^{L-1} \mathbb{E}_{q(z_{l+1}|s,x)}\left[KL\left[q(z_l|z_{l+1}, s, x)||p(z_l|z_{l+1}, x)\right]\right] \quad (8)$$

Plugging the simplifications obtained in Eq.7 and Eq.8 into Eq.6 we obtain the expression for the evidence lower bound given in Eq.2.