Appendix A

Blended Convolution and Synthesis for Efficient Discrimination of 3D Shapes

A. The derived Q_{nl} polynomials up to n=5, m=5:

Q_{nl} polynomials up to $n = 5$ and $m = 5$ are shown in Table 6.

Table 6: The derived Q_{nl} polynomials up to $n = 5$, $m = 5$.

Polynomial	Expression
Q_{00}	0
Q_{10}	$1. + 2x$
Q_{11}	$-1. - 1x$
Q_{20}	$-9.79 - 10.65x + 9x^2$
Q_{21}	$5.29 + 6.29x - 4x^2$
Q_{22}	$-1.99 - 3.63x + x^2$
Q_{30}	$-123.58 - 158.11x + 87.46x^2 + 32x^3$
Q_{31}	$70.26 + 89.41x - 50.31x^2 - 13.5x^3$
Q_{32}	$15.86 + 22.27x - 11.06x^2 - 0.5x^3$
Q_{33}	$-768.81 - 1006.25x + 512.65x^2 + 139.10x^3 + 104.16x^4$
Q_{40}	$-35.86 - 46.15x + 25.59x^2 + 4x^3$
Q_{41}	$422.87 + 550.70x - 287.81x^2 - 73.52x^3 - 42.66x^4$
Q_{42}	$-768.81 - 1014.25x + 480.65x^2 + 73.77x^3 + 13.5x^4$
Q_{43}	$-776.81 - 1034.25x + 454.65x^2 + 50.43x^3 - 2.66x^4$
Q_{44}	$-768.81 - 1022.25x + 464.65x^2 + 56.43x^3 + 0.16x^4$
Q_{50}	$-3683.18 - 4855.97x + 2342.20x^2 + 509.59x^3 + 340.36x^4 + 324x^5$
Q_{51}	$1960.80 + 2578.79x - 1263.64x^2 - 280.02x^3 - 167.77x^4 - 130.20x^5$
Q_{52}	$-981.80 - 1286.88x + 643.53x^2 + 141.74x^3 + 72.23x^4 + 42.66x^5$
Q_{53}	$463.12 + 604.69x - 309.13x^2 - 64.52x^3 - 25.87x^4 - 10.12x^5$
Q_{54}	$-208.26 - 272.17x + 140.81x^2 + 25.87x^3 + 7.44x^4 + 1.33x^5$
Q_{55}	$91.29 + 122.33x - 61.70x^2 - 9.53x^3 - 2.07x^4 - 0.04x^5$

B. Combined latent space projection and Volumetric Convolution with Roto-Translataional Kernels

Theorem 1: *Suppose $f, g : X \longrightarrow \mathbb{R}^3$ are square integrable complex functions defined in $\mathbb{B}^3$ so that $\langle f, f\rangle < \infty$ and $\langle g, g\rangle < \infty$. Further, suppose g is symmetric around north pole and $\tau(\alpha, \beta) = R_y(\alpha)R_z(\beta)$ where $R \in \mathbb{SO}(3)$ and $T_{r'}$ is translation of each point by r'. Then,*

$$\begin{aligned} f * g(r', \alpha, \beta) &:= \int_0^1 \int_0^{2\pi} \int_0^{\pi} P\{f(\theta, \phi, r)\}, T_{r'}\{\tau_{(\alpha,\beta)}(g(\theta, \phi, r))\} \sin\phi \, d\phi d\theta dr \\ &\approx \frac{4\pi}{3} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \sum_{l=0}^{n} \sum_{m=-l}^{l} \langle f_{nl}(r), Q_{n'l}(r)\rangle (e^{(n-l)r'} - e^{(n'-l)r'}) \hat{\Omega}_{n,l,m}(f) \hat{\Omega}_{n',l,0}(g) Y_{l,m}(\alpha, \beta), \end{aligned} \tag{13}$$

where $\hat{\Omega}_{n,l,m}(f), \hat{\Omega}_{n',l,0}(g)$ and $Y_{l,m}(\theta, \phi)$ are $(n, l, m)^{th}$ spectral moment of f, $(n', l, 0)^{th}$ spectral moment of g, and spherical harmonics function respectively. $P\{\cdot\}$ is the projection to a latent space, $\tau(\alpha, \beta) = R_y(\alpha)R_z(\beta)$ where $R \in \mathbb{SO}(3)$ and $T_{r'}$ is translation of each point by r'.

Proof: The input function f is projected to the latent space shape $\hat{f}$ by,

$$\hat{f}(\theta, \phi, r) = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{m=-l}^{l} \hat{\Omega}_{n,l,m}(f)\hat{Z}_{n,l,m}(\theta, \phi, r), \tag{14}$$

where spectral moment $\hat{\Omega}_{n,l,m}(f)$ can be obtained using,

$$\hat{\Omega}_{n,l,m}(f) = \int_0^1 \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi, r)\hat{Z}^{\dagger}_{n,l,m} r^2 \sin\phi \, dr d\phi d\theta. \tag{15}$$

and,

$$\hat{Z}_{n,l,m}(\theta, \phi, r) = \hat{Q}_{nl}(r)Y_{lm}(\theta, \phi), \tag{16}$$

where,

$$\hat{Q}_{nl}(r) = f_{nl}(r) - \sum_{k=0}^{n-1}\sum_{m=0}^{k} W_{nlkm}\hat{Q}_{km}(r), \tag{17}$$

$$f_{nl} = (-1)^l n \sum_{k=0}^{n} \frac{((n-l)r)^k}{k!}, \tag{18}$$

and,

$$Y_{l,m}(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\phi)e^{im\theta}, \tag{19}$$

where $\phi \in [0, \pi]$ is the polar angle, $\theta \in [0, 2\pi]$ is the azimuth angle, $l \in \mathbb{Z}^+$ is a non-negative integer, $m \in \mathbb{Z}$ is an integer, $|m| < l$, and $P_l^m(\cdot)$ is the associated Legendre function,

$$P_l^m(x) = (-1)^m \frac{(1-x^2)^{m/2}}{2^l l!} \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l. \tag{20}$$

In Eq. 17, the set $\{W_{nlkm}\}$ denotes trainable weights. Using this result, we can rewrite $f * g(r', \alpha, \beta)$ as,

$$\begin{aligned} f * g(r, \alpha, \beta) = \langle &\sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{m=-l}^{l} \hat{\Omega}_{n,l,m}(f)\hat{Z}_{n,l,m}(\theta, \phi, r), \\ &T_{r'}\{\tau_{(\alpha,\beta)}(\sum_{n'=0}^{\infty}\sum_{l'=0}^{n}\sum_{m'=-l'}^{l'} \hat{\Omega}_{n',l',m'}(g)\hat{Z}_{n',l',m'}(\theta, \phi, r)))\}\rangle_{\mathbb{B}^3} \end{aligned} \tag{21}$$

Using the properties of inner product, this can be rewritten as,

$$\begin{aligned} f * g(r', \alpha, \beta) = &\sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{m=-l}^{l}\sum_{n'=0}^{\infty}\sum_{l'=0}^{'n}\sum_{m'=-l'}^{l'} \hat{\Omega}_{n,l,m}(f)\hat{\Omega}_{n',l',m'}(g) \\ &\langle \hat{Z}_{n,l,m}(\theta, \phi, r), T_{r'}\{\tau_{(\alpha,\beta)}(\hat{Z}_{n',l',m'}(\theta, \phi, r))\}\rangle_{\mathbb{B}^3} \end{aligned} \tag{22}$$

Consider the term,

$$\begin{aligned} \Gamma &= \langle \hat{Z}_{n,l,m}(\theta, \phi, r), T_{r'}\{\tau_{(\alpha,\beta)}(\hat{Z}_{n',l',m'}(\theta, \phi, r)))\}\rangle_{\mathbb{B}^3} \\ &= \langle \hat{Q}_{nl}(r)Y_{lm}(\theta, \phi), T_{r'}\{\tau_{(\alpha,\beta)}(\hat{Q}_{n'l'}(r)Y_{l'm'}(\theta, \phi))\}\rangle_{\mathbb{B}^3} \end{aligned} \tag{23}$$

Γ can be decomposed into its angular and linear components as,

$$\Gamma = \int_0^1 \hat{Q}_{nl}(r)T_{r'}\{\hat{Q}_{n'l'}(r)\}r^2 dr \int_0^{2\pi}\int_0^{\pi} Y_{lm}(\theta, \phi)\tau_{(\alpha,\beta)}(Y_{l'm'}(\theta, \phi))\} sin\phi d\phi d\theta. \tag{24}$$

First, consider the angular component,

$$Ang(\Gamma) = \int_0^{2\pi}\int_0^{\pi} Y_{lm}(\theta,\phi)\tau_{(\alpha,\beta)}(Y_{l'm'}(\theta,\phi))\} sin\phi d\phi d\theta. \tag{25}$$

Since $g(\theta,\phi,r)$ is symmetric around y, using the properties of spherical harmonics, Eq. 25 can be rewritten as,

$$Ang(\Gamma) = \int_0^{2\pi}\int_0^{\pi} Y_{lm}(\theta,\phi)\sum_{m''=-l'}^{l'} Y_{l',m''}D^{l'}_{m''0}(\alpha,\beta)\} sin\phi d\phi d\theta \tag{26}$$

where $D^{l'}_{mm'}$ is the Wigner-D matrix. But $D^{l'}_{m''0} = Y_{l',m''}$, and hence,

$$Ang(\Gamma) = \sum_{m''=-l'}^{l'} Y_{l',m''}(\alpha,\beta)\int_0^{2\pi}\int_0^{\pi} Y_{lm}(\theta,\phi)Y_{l',m''}(\theta,\phi)\} sin\phi d\phi d\theta \tag{27}$$

Since spherical harmonics are orthogonal,

$$Ang(\Gamma) = C_{ang}Y_{l,m}(\alpha,\beta), \tag{28}$$

where C_{ang} is a constant. Consider the linear component of Eq. 24. It is important to note that for simplicity, we derive equations for the orthogonal case and use the same results for non-orthogonal case. In practice, this step does not reduce accuracy.

$$Lin(\Gamma) = \int_0^1 \hat{Q}_{nl}(r)T_{r'}\{\hat{Q}_{n'l'}(r)\}r^2 dr. \tag{29}$$

Since $\hat{Q}_{nl}(r)$ is a linear combination of $f_{nl} \approx (-1)^l n\exp(r(n-l))$, it is straightforward to see that,

$$Q_{nl}(r+r') = f_{nl}(r)\exp((n-l)r') - \sum_{k=0}^{n-1}\sum_{m=0}^{k} C_{nlkm}Q_{km}(r)\exp(k-m)r'. \tag{30}$$

Also, we have derived that $l = l'$ from the result in Eq. 28. Applying this result and Eq. 30 to Eq. 29 gives,

$$\langle Q_{nl}(r+r'), Q_{n'l}(r)\rangle = \langle f_{nl}(r+r'), Q_{n'l}(r)\rangle - \sum_{k=0}^{n-1}\sum_{m=0}^{k} C_{nlkm}\langle Q_{km}(r+r'), Q_{n'l}(r)\rangle, \tag{31}$$

$$\langle Q_{nl}(r+r'), Q_{n'l}(r)\rangle = \langle f_{nl}(r), Q_{n'l}(r)\rangle e^{(n-l)r'} - \sum_{k=0}^{n-1}\sum_{m=0}^{k} C_{nlkm}\langle Q_{km}(r)e^{(k-m)r'}, Q_{(n'l}(r)\rangle. \tag{32}$$

Since Q_{km} and $Q_{n'l}$ are orthogonal,

$$\langle Q_{nl}(r+r'), Q_{n'l}(r)\rangle = \langle f_{nl}(r), Q_{n'l}(r)\rangle e^{(n-l)r'} - C_{nln'l}e^{(n'-l)r'}||Q_{n'l}||^2. \tag{33}$$

But since for orthogonal case, $C_{nln'l'} = \frac{<f_{nl}, Q_{n'l'}>}{||Q_{n'l'}||^2}$,

$$\langle Q_{nl}(r+r'), Q_{n'l}(r)\rangle = \langle f_{nl}(r), Q_{n'l}(r)\rangle e^{(n-l)r'} - \langle f_{nl}(r), Q_{n'l'}(r)\rangle e^{(n'-l)r'}, \tag{34}$$

$$\langle Q_{nl}(r+r'), Q_{n'l}(r)\rangle = \langle f_{nl}(r), Q_{n'l}(r)\rangle (e^{(n-l)r'} - e^{(n'-l)r'}). \tag{35}$$

Combining Eq. 28 and Eq. 35 for Eq. 22 and choosing the normalization constant to be $\frac{4\pi}{3}$ (since the integration is over unit ball) gives,

$$f * g(r',\alpha,\beta) \approx \frac{4\pi}{3}\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\sum_{l=0}^{n}\sum_{m=-l}^{l}\langle f_{nl}(r), Q_{n'l}(r)\rangle(e^{(n-l)r'} - e^{(n'-l)r'})\hat{\Omega}_{n,l,m}(f)\hat{\Omega}_{n',l,0}(g)Y_{l,m}(\alpha,\beta). \tag{36}$$

Q.E.D.

Table 7: Ablation study on the input point cloud density. We sample the input points on a grid (r= 25, θ = 36, ϕ = 18) before feeding to the network.

Original point cloud sampling	**Accuracy**
(r= 250, θ = 200, ϕ = 200)	94.22%
(r= 300, θ = 250, ϕ = 250)	94.21%
(r= 400, θ = 300, ϕ = 300)	**94.23%**
(r= 500, θ = 400, ϕ = 400)	94.20%

C. Ablation study on input point cloud density

A critical problem associated with directly consuming point clouds, in order to learn features, is the redundancy of information. This property hampers optimal feature learning using neural network based models, by imposing an additional overhead. To verify this, we conduct an ablation study on the density of the input point cloud, and observe the performance variations of our model. The obtained results are reported in Table 7. As the results suggest, there is no clear variation of classification performance, although the input sampling density is increased. Therefore, it can be empirically concluded that input point clouds are not optimal to be directly fed to learning networks, due to their inherent redundancy. As a result, significant reduction in their density could still lead to comparable performance with that of the original point cloud.