

# Supplementary Material: Shape-Tailored Local Descriptors and their Application to Segmentation and Tracking

Naeemullah Khan<sup>1</sup>, Marei Algarni<sup>1</sup>, Anthony Yezzi<sup>2</sup>, and Ganesh Sundaramoorthi<sup>1</sup>

<sup>1</sup>King Abdullah University of Science & Technology (KAUST), Saudi Arabia

<sup>2</sup> School of Electrical & Computer Engineering, Georgia Institute of Technology, USA

`{naeemullah.khan, marei.algarni, ganesh.sundaramoorthi}@kaust.edu.sa, ayezzi@ece.gatech.edu`

## 1. Computations for Propositions and Lemmas

We detail the calculations that were left out in Section 2.2 of the manuscript. We repeat some definitions for the convenience of the reader. Note the Lemma and Proposition numbers correspond to the numbers in the manuscript, however, equation numbers do not correspond.

**Definition 1** (Shape-Tailored Local Descriptors). *Let  $R \subset \mathbb{R}^2$  be a bounded region with non-zero area and smooth boundary  $\partial R$ . Let  $I : R \rightarrow \mathbb{R}^k$ . A **Shape-Tailored Descriptor**,  $\mathbf{u} : R \rightarrow \mathbb{R}^M$  (where  $M = n \times m$ ,  $n, m \geq 1$ ) consists of components  $u_{ij} : R \rightarrow \mathbb{R}$  so that  $\mathbf{u} = (u_{11}, \dots, u_{1m}, \dots, u_{n1}, \dots, u_{nm})^T$ . The components are defined as:*

$$\begin{cases} u_{ij}(x) - \alpha_i \Delta u_{ij}(x) = J_j(x) & x \in R \\ \nabla u_{ij}(x) \cdot N = 0 & x \in \partial R \end{cases}, \quad (1)$$

where  $1 \leq i \leq n$ ,  $1 \leq j \leq m$ ,  $\Delta$  denotes the Laplacian,  $\nabla$  denotes the gradient,  $N$  is the unit outward normal to  $R$ ,  $\alpha_i > 0$  are scales, and  $J_j : R \rightarrow \mathbb{R}$  are point-wise functions of the image  $I$ . In vector form, this is equivalent to

$$\begin{cases} \mathbf{u}(x) - A \Delta \mathbf{u}(x) = \mathbf{J}(x) & x \in R \\ D \mathbf{u}(x) N = \mathbf{0} & x \in \partial R \end{cases}, \quad (2)$$

where  $A = \text{diag}(\alpha_1 \mathbf{1}_{1 \times m}, \dots, \alpha_n \mathbf{1}_{1 \times m})$  (an  $M \times M$  diagonal matrix),  $\mathbf{1}_{1 \times m}$  is a  $1 \times m$  matrix of ones,  $D$  denotes the spatial derivative operator, and  $\mathbf{J} = (J_1, \dots, J_m, \dots, J_1, \dots, J_m, \dots)^T$ .

**Lemma 1** (PDE for Descriptor Variation). *Let  $u$  satisfy the PDE (1),  $h$  be a perturbation of  $\partial R$ , and  $u_h$  denote the variation of  $u$  with respect to the perturbation  $h$ . Then*

$$\begin{cases} u_h(x) - \alpha_i \Delta u_h(x) = 0 & x \in R \\ \nabla u_h(x) \cdot N = u_s(x)(h_s \cdot N) - N^T H u(x) \cdot h & x \in \partial R \end{cases} \quad (3)$$

where  $s$  is the arc-length parameter of  $\partial R$ ,  $h_s$  denotes the derivative with respect to arc-length, and  $H u(x)$  denotes the Hessian matrix.

*Proof.* The PDE for  $u_h$  is obtained by computing the variation with respect to  $h$  of both conditions of the PDE (1). The variation of the first equation in (1) leads to the first equation in (3). This is because the variation and spatial derivatives commute by equality of mixed partials as the variation and spatial derivative operators are independent. Next we compute the variation of the boundary condition using the Chain Rule:

$$d[\nabla u(c(p)) \cdot N] \cdot h = N^T H u(c(p)) \cdot h + \nabla u(c(p)) \cdot N_h = 0, \quad (4)$$

where again we have switched the order of spatial derivatives and the variation by equality of mixed partials. Let  $c$  be a parameterization of  $\partial R$  with parameter  $p$ , and  $c_p$  indicate the derivative w.r.t the parameter. The variation of  $N = JT = Jc_p/|c_p|$  ( $J$  is a  $90^\circ$  rotation matrix) is

$$N_h = J \frac{h_p |c_p| - \frac{c_p \cdot h_p}{|c_p|} c_p}{|c_p|^2} = J(h_s - (h_s \cdot T)T) = -(h_s \cdot N)T. \quad (5)$$

Substituting (5) into (4) leads to the boundary condition in (3).  $\square$

**Definition 2** (Green's Function for (3)). *The Green's function,  $K_{\alpha_i} : R \times R \rightarrow \mathbb{R}$ , for the problem (3) (and (1)) satisfies*

$$\begin{cases} K_{\alpha_i}(x, y) - \alpha_i \Delta_x K_{\alpha_i}(x, y) = \delta(x - y) & x, y \in R \\ \nabla_x K_{\alpha_i}(x, y) \cdot N = 0 & x \in \partial R, y \in R \end{cases} \quad (6)$$

where  $\Delta_x$  ( $\nabla_x$ ) is the Laplacian (gradient) with respect to  $x$ , and  $\delta$  is the Delta function.

**Lemma 2** (Region and Boundary Integrals of  $K$ ). *If  $f : R \rightarrow \mathbb{R}$ ,  $g : \partial R \rightarrow \mathbb{R}$  and*

$$\hat{u}(x) = \int_R K_{\alpha_i}(x, y) f(y) dy - \int_{\partial R} K_{\alpha_i}(x, y) g(y) ds(y) \quad (7)$$

where  $K$  is the Green's function that satisfies (6), then  $\hat{u}$  satisfies

$$\begin{cases} \hat{u}(x) - \alpha_i \Delta \hat{u}(x) = f(x) & x \in R \\ \nabla \hat{u}(x) \cdot N = g(x) & x \in \partial R \end{cases}. \quad (8)$$

*Proof.* This is a standard result in PDE (see, for example [2], for a proof).  $\square$

**Proposition 1** (Descriptor Gradient). *The gradient with respect to  $c = \partial R$  of  $u(x)$  (one component of  $\mathbf{u}(x)$ ), which satisfies the PDE (1), is*

$$\nabla_c u(x) = \left[ (Du)^T D_y K_{\alpha_i}(x, \cdot) + \frac{1}{\alpha_i} K_{\alpha_i}(x, \cdot) (u - J_j) \right] N \quad (9)$$

where  $N$  is the outward normal,  $D_y$  denotes the derivative wrt the second argument of  $K_{\alpha_i}$ , and  $Du$  indicates the spatial derivative of  $u$ .

*Proof.* By the property (7), we may express the solution of (3) as

$$\begin{aligned} u_h(x) &= - \int_{\partial R} K_{\alpha_i}(x, y) [u_s(y) h_s \cdot N - N^T H u(x) \cdot h] ds(y) \\ &= \int_{\partial R} [\partial_s (K_{\alpha_i} u_s N) + K_{\alpha_i} N^T H u] \cdot h ds, \end{aligned}$$

where  $\partial_s$  denotes derivative with respect to arc-length. Therefore,  $\nabla_c u(x)$  is the bracketed expression above, which we now simplify. We note that  $N_s = \kappa T$ , and by differentiating the boundary condition  $\nabla u(c(s)) \cdot N = 0$  in  $s$ , we find that  $N^T H u(x) \cdot T = -u_s \kappa$  where  $\kappa$  is the signed curvature of  $c$ . Using the former two properties, we have

$$\begin{aligned} \nabla_c u(x) &= \partial_s (K_{\alpha_i} u_s) N + K_{\alpha_i} u_s \kappa T + K_{\alpha_i} (N^T H u \cdot N) N + K_{\alpha_i} (N^T H u \cdot T) T \\ &= [\partial_s (K_{\alpha_i} u_s) + K_{\alpha_i} (N^T H u \cdot N)] N. \end{aligned} \quad (10)$$

Now note that  $u_{ss} = \partial_s (\nabla u \cdot T) = T^T H u \cdot T + \nabla u \cdot \kappa N = T^T H u \cdot T$  using that  $\nabla u \cdot N = 0$ . This implies that  $\Delta u = u_{ss} + N^T H u \cdot N$ . Using (10), we have that

$$\nabla_c u(x) = [K_{\alpha_i, s} u_s + K_{\alpha_i} \Delta u] N. \quad (11)$$

Using  $\nabla u \cdot N = 0$  on  $\partial R$  and  $u - \alpha_i \Delta u = J_j$  into (11) gives

$$\nabla_c u(x) = \left[ \nabla u \cdot \nabla_y K_{\alpha_i}(x, \cdot) + \frac{1}{\alpha_i} (u - J_j) K_{\alpha_i}(x, \cdot) \right] N. \quad (12)$$

$\square$

**Proposition 2** (Integrals of Descriptor Gradient). *Let  $\mathbf{f}, \mathbf{g} : R \rightarrow \mathbb{R}^M$  and  $\mathbf{u}$  be the Shape-Tailored Descriptor in  $R$  (as in (2)). Then*

$$\mathbb{I}_d[R, \mathbf{u}, \mathbf{f}, \mathbf{g}] := - \int_{\partial R} \nabla_c \mathbf{u}(x) \mathbf{g}(x) \, ds(x) + \int_R \nabla_c \mathbf{u}(x) \mathbf{f}(x) \, dx = (\text{tr}[(D\mathbf{u})^T D\hat{\mathbf{u}}] + (\mathbf{u} - \mathbf{J})^T A^{-1} \hat{\mathbf{u}}) N \quad (13)$$

where  $dx$  is the area measure,  $ds$  is the arclength measure,  $N$  is the outward normal to the boundary of  $R$ ,  $\text{tr}$  denotes matrix trace, and

$$\begin{cases} \hat{\mathbf{u}}(x) - A\Delta\hat{\mathbf{u}}(x) = \mathbf{f}(x) & x \in R \\ D\hat{\mathbf{u}}(x)N = \mathbf{g}(x) & x \in \partial R \end{cases}. \quad (14)$$

*Proof.* Let  $u, f, g$  denote components of  $\mathbf{u}, \mathbf{f}, \mathbf{g}$ . Then

$$\int_{\partial R} \nabla_c u(x) g(x) \, ds(x) = \int_{\partial R} N \left[ (Du)^T D_y K_{\alpha_i}(x, \cdot) + \frac{1}{\alpha_i} K_{\alpha_i}(x, \cdot) (u - J_j) \right] g(x) \, ds(x) \quad (15)$$

$$= N \left[ \nabla_y \cdot \int_{\partial R} K_{\alpha_i}(x, \cdot) (Du)^T g(x) \, ds(x) + \frac{1}{\alpha_i} \int_{\partial R} K_{\alpha_i}(x, \cdot) (u - J_j) g(x) \, ds(x) \right] \quad (16)$$

$$= N \left[ \nabla_y \cdot (Du)^T \int_{\partial R} K_{\alpha_i}(x, \cdot) g(x) \, ds(x) + \frac{1}{\alpha_i} (u - J_j) \int_{\partial R} K_{\alpha_i}(x, \cdot) g(x) \, ds(x) \right]. \quad (17)$$

Note that  $\nabla_y \cdot$  indicates divergence with respect to the second argument of the kernel  $K_{\alpha_i}$ , and also that the arguments of  $u, Du, J_j$  depend on the point of the curve that has been suppressed for ease of notation. We may follow a similar computation to arrive at

$$\int_R \nabla_c u(x) f(x) \, dx = N \left[ \nabla_y \cdot (Du)^T \int_R K_{\alpha_i}(x, \cdot) f(x) \, dx + \frac{1}{\alpha_i} (u - J_j) \int_R K_{\alpha_i}(x, \cdot) f(x) \, dx \right]. \quad (18)$$

Then by summing expressions, we arrive at

$$- \int_{\partial R} \nabla_c u(x) g(x) \, ds(x) + \int_R \nabla_c u(x) f(x) \, dx = N \left[ (Du)^T D\hat{u} + \frac{1}{\alpha_i} (u - J_j) \hat{u} \right] \quad (19)$$

where

$$\hat{u}(y) = - \int_{\partial R} K_{\alpha_i}(x, y) g(x) \, ds(x) + \int_R K_{\alpha_i}(x, y) f(x) \, dx, \quad (20)$$

by symmetry of the Green's function and Lemma 2. Writing (19) and (20) in vector form gives the result of this proposition.  $\square$

**Proposition 3** (Weighted Area Gradient). *Let  $F : \mathbb{R}^M \rightarrow \mathbb{R}$  and  $\mathbf{u} : R \rightarrow \mathbb{R}^M$  be the Shape-Tailored Descriptor on  $R$ . Define the weighted area functionals as  $A_F = \int_R F(\mathbf{u}(x)) \, dx$ . Then*

$$\nabla_c A_F = (F \circ \mathbf{u})N + \mathbb{I}_d[R, \mathbf{u}, (\nabla F) \circ \mathbf{u}, \mathbf{0}] \quad (21)$$

where  $\mathbb{I}_d$  is defined as in Proposition 2.

*Proof.* The gradient above can be derived by using the Chain-Rule. The gradient of the functional, assuming that the descriptor does not vary with the curve, is added to the gradient of the functional with respect to the descriptor. The former is obtained using classical results (e.g., [6]), and is the first term in (21). The latter is  $\int_R \nabla_c \mathbf{u}(x) \nabla F(\mathbf{u}(x)) \, dx$ , which by Proposition 2 results in the second term of (21).  $\square$

## 2. Numerical Discretization

We show the discretization scheme for the PDE

$$\begin{cases} u(x) - \alpha \Delta u(x) = f(x) & x \in R \\ \nabla u(x) \cdot N = g(x) & x \in \partial R \end{cases}, \quad (22)$$

where  $N$  is the outward normal. Note that this is the type of equations that are satisfied by the descriptors  $\mathbf{u}, \mathbf{v}$  and the functions  $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ . We assume that  $R = \{x \in \Omega : \Psi(x) \leq 0\}$  where  $\Psi : \Omega \rightarrow \mathbb{R}$  is the level set function. We use central differences to discretize the Laplacian:

$$\Delta u(x) = \sum_{y \in N_x} (u(y) - u(x)) = \sum_{y \in N_x \cap R} (u(y) - u(x)) + \sum_{y \in N_x \cap R^c} (u(y) - u(x)) \quad (23)$$

where  $N_x$  is a 4-neighbor of  $x$ . Note that  $u(y)$  for  $y \in R^c$  is not defined, but using a discretization of the boundary condition, we have that  $u(y) - u(x) = g(x)$  for  $y \in N_x \cap R^c$  and  $x \in R$ . Thus, we have

$$\Delta u(x) = \sum_{y \in N_x \cap R} (u(y) - u(x)) + \sum_{y \in N_x \cap R^c} g(x). \quad (24)$$

Therefore, the final discretization of the PDE is

$$(1 + \alpha |N_x \cap R|)u(x) - \sum_{y \in N_x \cap R} u(y) = f(x) + |N_x \cap R^c|g(x), \quad x \in R \quad (25)$$

where  $|N_x \cap R|$  is the number of pixels in  $N_x \cap R$ . This is now in the form where standard linear solvers (e.g., conjugate gradient, multigrid) may be applied.

Note that in narrowband level set methods, the speed function must be extended into the narrowband (1-pixel dilation of  $R$ ), and this requires that  $\mathbf{u}, \hat{\mathbf{u}}$  be extended into the narrowband. Therefore, we show how  $u$  defined in (22) can be extended to the narrowband. This can be accomplished by discretizing the boundary condition, which yields

$$u(y) = u(x) + g(x), \quad y \in N_x \cap R^c \quad (26)$$

and  $x \in N_y \cap R$  is such that  $x$  is the point with closest distance to a zero crossing of the level set function  $\Psi$ .

### 3. More Results from Experiments on Real Texture Dataset

The next figures show more results of segmentation on the Real Texture Segmentation Dataset.

### References

- [1] P. Arbelaez, M. Maire, C. Fowlkes, and J. Malik. Contour detection and hierarchical image segmentation. *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, 33(5):898–916, 2011. [5](#), [6](#), [7](#), [8](#), [9](#)
- [2] L. C. Evans. Partial differential equations. 1998. [2](#)
- [3] B.-W. Hong, S. Soatto, K. Ni, and T. Chan. The scale of a texture and its application to segmentation. In *Computer Vision and Pattern Recognition, 2008. CVPR 2008. IEEE Conference on*, pages 1–8. IEEE, 2008. [5](#), [6](#), [7](#), [8](#), [9](#)
- [4] P. Isola, D. Zoran, D. Krishnan, and E. H. Adelson. Crisp boundary detection using pointwise mutual information. In *Computer Vision–ECCV 2014*, pages 799–814. Springer, 2014. [5](#), [6](#), [7](#), [8](#), [9](#)
- [5] K. Ni, X. Bresson, T. Chan, and S. Esedoglu. Local histogram based segmentation using the wasserstein distance. *International Journal of Computer Vision*, 84(1):97–111, 2009. [5](#), [6](#), [7](#), [8](#), [9](#)
- [6] S. C. Zhu and A. Yuille. Region competition: Unifying snakes, region growing, and bayes/mdl for multiband image segmentation. *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, 18(9):884–900, 1996. [3](#)





Figure 1. Sample Results on the Real Texture Dataset. Segmentation boundaries are displayed for various methods.



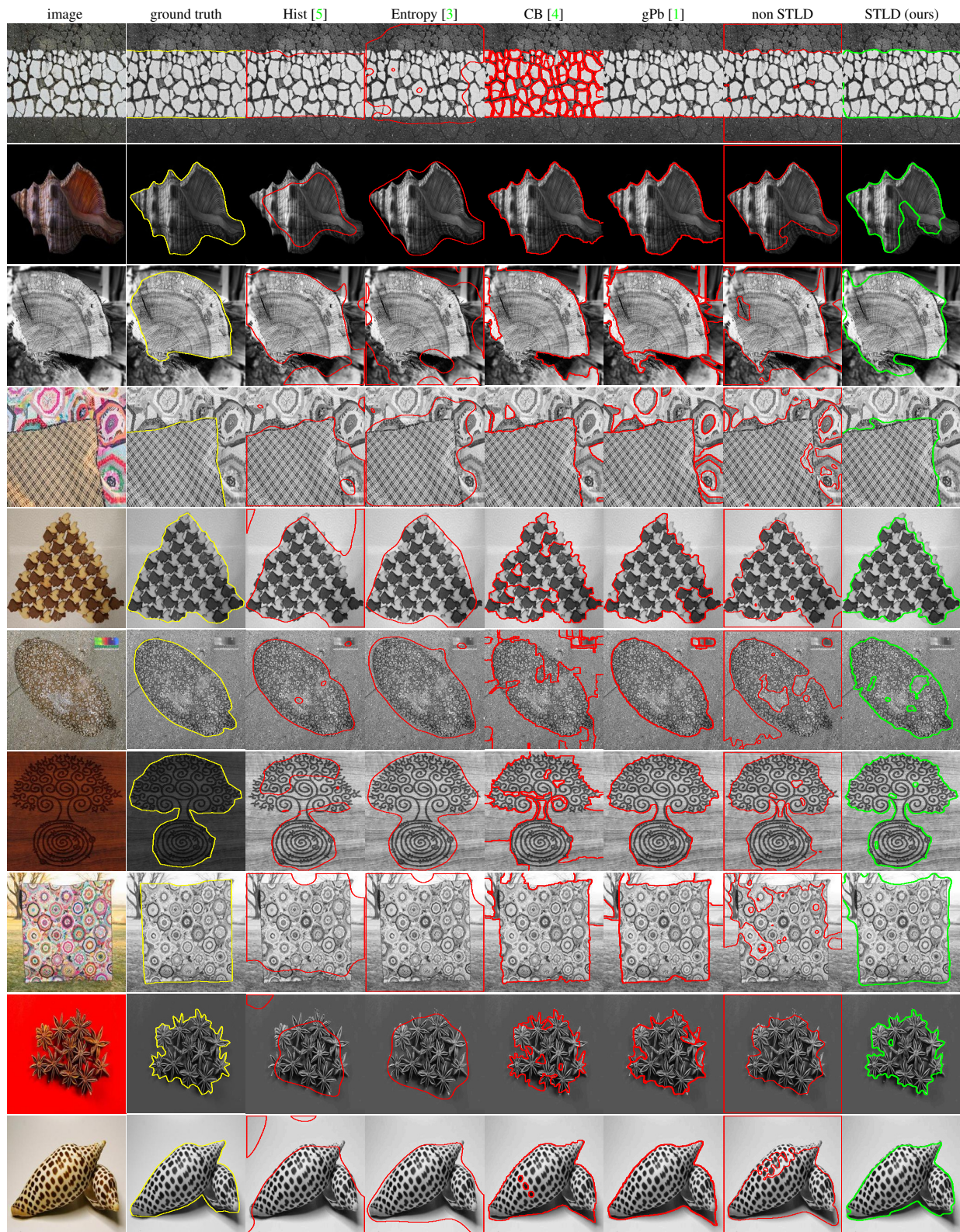


Figure 2. Sample Results on the Real Texture Dataset. Segmentation boundaries are displayed for various methods.





Figure 3. Sample Results on the Real Texture Dataset. Segmentation boundaries are displayed for various methods.



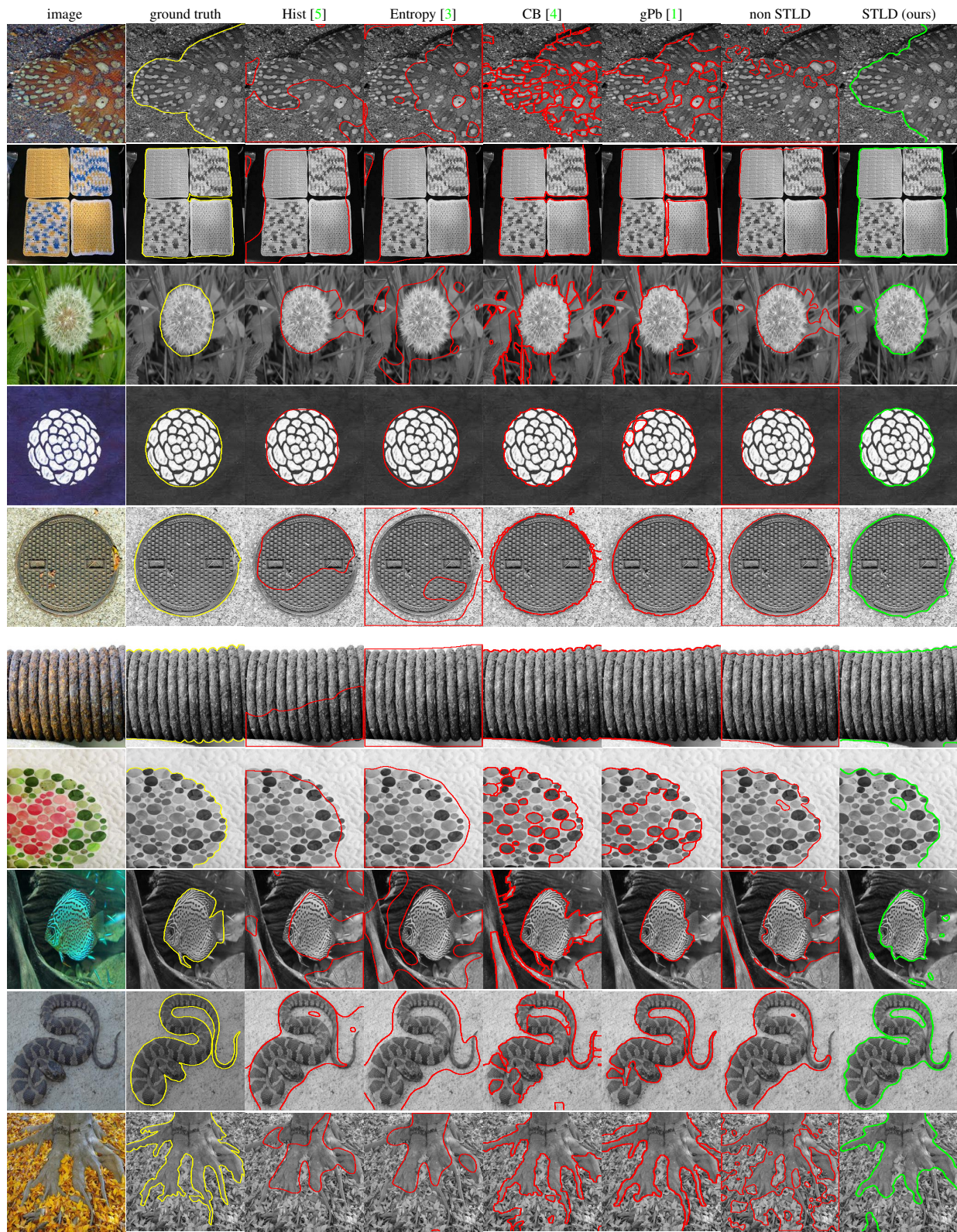


Figure 4. Sample Results on the Real Texture Dataset. Segmentation boundaries are displayed for various methods.



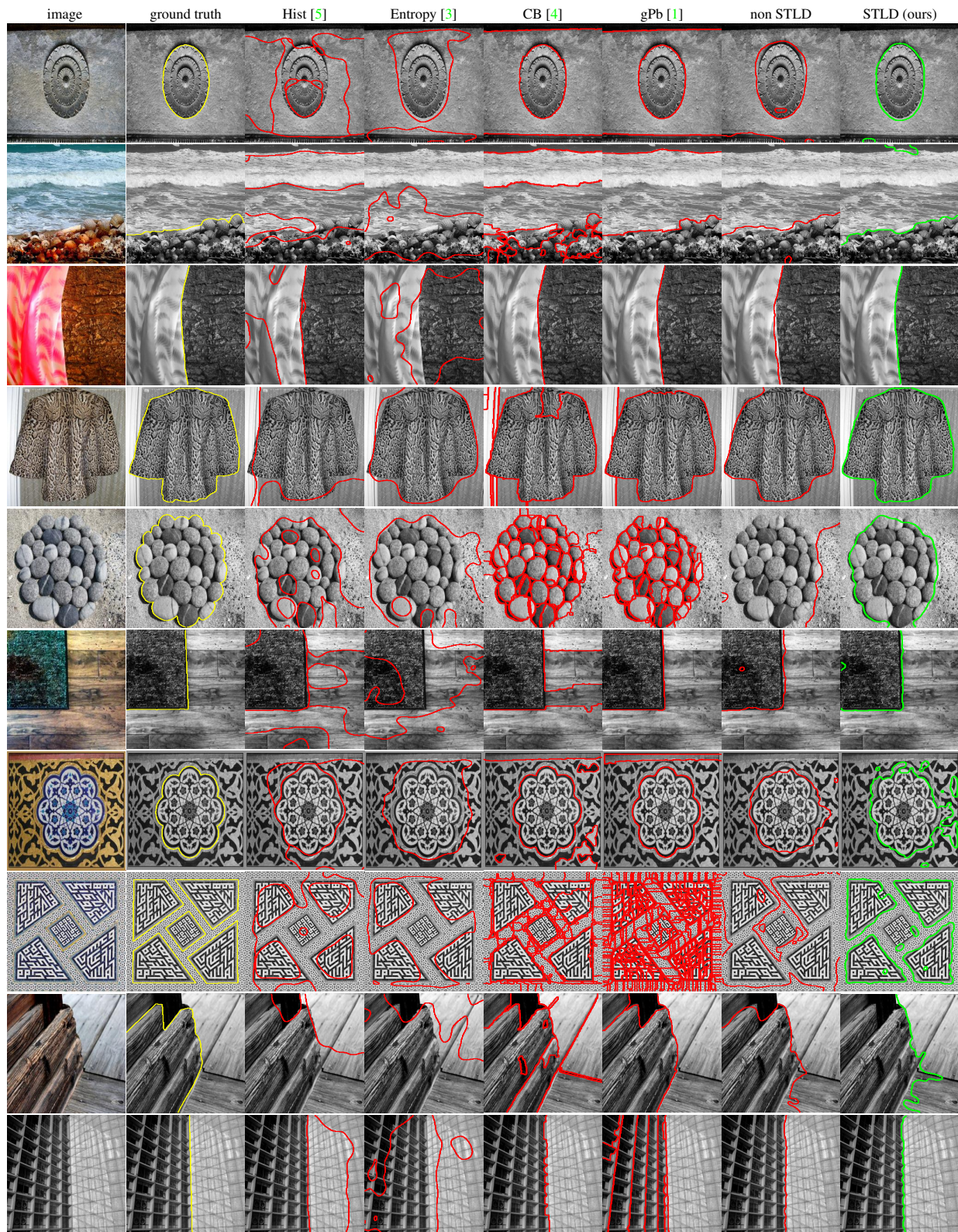


Figure 5. Sample Results on the Real Texture Dataset. Segmentation boundaries are displayed for various methods.