GRSA: Generalized Range Swap Algorithm for the Efficient Optimization of MRFs

1. Proof of Theorem 1

Before proving Theorem 1, we first give the following lemmas and the definition of *submodular set*.

Lemma 1. *For $b_1, b_2 > 0$, the following conclusion holds.*

$$\frac{a_1}{b_1} \geq \frac{a_2}{b_2} \Leftrightarrow \frac{a_1}{b_1} \geq \frac{a_1+a_2}{b_1+b_2} \geq \frac{a_2}{b_2}. \quad (1)$$

The proof is straightforward and we omit it.

Lemma 2. *Assuming that function $g(x)$ is convex on $[a, b]$ and there are three points $x_1, x, x_2 \in [a, b]$ satisfying $x_1 > x > x_2$, there is*

$$\frac{g(x_1)-g(x)}{x_1-x} \geq \frac{g(x_1)-g(x_2)}{x_1-x_2} \geq \frac{g(x)-g(x_2)}{x-x_2}. \quad (2)$$

Proof. Since $x_1 > x > x_2$, there exists $\lambda \in (0, 1)$ satisfying $x = (1-\lambda)x_1 + \lambda x_2$. Then by the definition of convex function, there is $(1-\lambda)g(x_1) + \lambda g(x_2) \geq g(x)$ and thus

$$(1-\lambda)(g(x_1)-g(x)) \geq \lambda(g(x)-g(x_2)) \quad (3)$$

Considering that $x_1 > x_2$ and $0 < \lambda < 1$, we can divide $\lambda(1-\lambda)(x_1-x_2)$ on both sides of (3) and obtain

$$\frac{g(x_1)-g(x)}{\lambda(x_1-x_2)} \geq \frac{g(x)-g(x_2)}{(1-\lambda)(x_1-x_2)} \quad (4)$$

$$\frac{g(x_1)-g(x)}{x_1-x} \geq \frac{g(x)-g(x_2)}{x-x_2}. \quad (5)$$

At last, the conclusion (2) can be proved by applying Lemma 1 to (5). □

Lemma 3. *Assuming that $g(x)$ is convex on $[a, b]$ and there are four points $x_1, x_2, x_3, x_4 \in [a, b]$ satisfying $x_1 > x_3 \geq x_4$ and $x_1 \geq x_2 > x_4$, there is*

$$\frac{g(x_1)-g(x_3)}{x_1-x_3} \geq \frac{g(x_2)-g(x_4)}{x_2-x_4}. \quad (6)$$

Proof. 1. If $x_2 = x_3$, the conclusion is straightforward by Lemma 2.

2. If $x_2 > x_3$, there is $x_1 \geq x_2 > x_3 \geq x_4$. We first consider the case where $x_1 > x_2 > x_3 > x_4$ and can use Lemma 2 to obtain (6) by

$$\frac{g(x_1)-g(x_3)}{x_1-x_3} \geq \frac{g(x_2)-g(x_3)}{x_2-x_3} \geq \frac{g(x_2)-g(x_4)}{x_2-x_4}.$$

For the case where $x_1 = x_2 > x_3 > x_4$, the right half of the above inequality holds and we can obtain (6) by replacing x_2 with x_1. The case where $x_1 > x_2 > x_3 = x_4$ can be obtained in a similar way while the case where $x_1 = x_2 > x_3 = x_4$ is straightforward.

3. If $x_2 < x_3$, there is $x_1 > x_3 > x_2 > x_4$. Using Lemma 2, we can obtain (6) by

$$\frac{g(x_1)-g(x_3)}{x_1-x_3} \geq \frac{g(x_3)-g(x_2)}{x_3-x_2} \geq \frac{g(x_2)-g(x_4)}{x_2-x_4}$$

Therefore, the proof of Lemma 3 is competed. □

Lemma 4. *Given a function $g(x)$ $(x = |\alpha-\beta|)$ on domain $X = [0, c]$, assume $g(x)$ is locally convex on interval $X_s = [a, b]$ $(0 \leq a < b \leq c)$, and it satisfies $a\{g(a+1)-g(a)\} \geq g(a)-g(0)$. Then we have*

$$\frac{g(x_1)-g(x_3)}{x_1-x_3} \geq \frac{g(x_2)-g(0)}{x_2} \quad (7)$$

where x_1, x_2, $x_3 \in X_s$ where $x_3 < x_1$ and $x_2 < x_1$.

Proof. Since $x_1 > x_3 \geq a$ and $x_1 \in \mathbb{N}$, we have $x_1 \geq a+1 > a$. Then considering that $x_1 > x_3 \geq a$, we can use Lemma 3 to obtain

$$\frac{g(x_1)-g(x_3)}{x_1-x_3} \geq \frac{g(a+1)-g(a)}{a+1-a} \geq \frac{g(a)-g(0)}{a}, \quad (8)$$

where the second inequality comes from $a\{g(a+1)-g(a)\} \geq g(a)-g(0)$.

If $x_2 = a$, the conclusion is obtained from (8). Otherwise, there is $x_1 > x_2 > a$ and $x_1 > x_3 \geq a$. Using Lemma 3, we obtain

$$\frac{g(x_1)-g(x_3)}{x_1-x_3} \geq \frac{g(x_2)-g(a)}{x_2-a}. \quad (9)$$

Combining (8) and (9), we can obtain

$$\begin{aligned}\frac{g(x_1)-g(x_3)}{x_1-x_3} &\geq \max\left\{\frac{g(x_2)-g(a)}{x_2-a}, \frac{g(a)-g(0)}{a}\right\}\\ &\geq \frac{g(x_2)-g(a)+g(a)-g(0)}{x_2-a+a}\\ &= \frac{g(x_2)-g(0)}{x_2},\end{aligned}$$

where the second inequality is due to Lemma 1 and this completes the proof. □

Definition 1. *Given a pairwise potential $\theta(\alpha,\beta)$, we call $\mathcal{L}_s$ a submodular set, if it satisfies*

$$\theta(l_{i+1},l_j)-\theta(l_{i+1},l_{j+1})-\theta(l_i,l_j)+\theta(l_i,l_{j+1}) \geq 0 \qquad (10)$$

for any pair of labels l_i, $l_j \in \mathcal{L}_s (1 \leq i,j < m)$.

Theorem 1. *Given a pairwise function $\theta(\alpha,\beta)=g(x)$ $(x=|\alpha-\beta|)$ on domain $X=[0,c]$, assume there is an interval $X_s=[a,b]$ $(0 \leq a < b \leq c)$ satisfying: (i) $g(x)$ is locally convex on $[a,b]$, and (ii) $a\{g(a+1)-g(a)\} \geq g(a)-g(0)$. Then $\mathcal{L}_s=\{l_1,\cdots,l_m\}$ is a* ***submodular subset****, if $|l_i-l_j| \in [a,b]$ for any pair of labels $l_i, l_j \in \mathcal{L}_s$.*

Proof. Since $\theta(\alpha,\beta)$ is semimetric and satisfies $\theta(\alpha,\beta)=\theta(\beta,\alpha)$, we only consider $l_i, l_{i+1}, l_j, l_{j+1} \in \mathcal{L}_s$ where $i \geq j$. Let

$$\begin{aligned} x_1 &= l_{i+1}-l_j, & x_2 &= l_{i+1}-l_{j+1},\\ x_3 &= l_i-l_j, & x_4 &= l_i-l_{j+1}\end{aligned}$$

For $i \geq j$, we have $x_i > x_j$, and $x_1-x_2=x_3-x_4$. We can define

$$\lambda = \frac{x_3-x_4}{x_1-x_4} = \frac{x_1-x_2}{x_1-x_4}, \quad (0<\lambda<1) \qquad (11)$$

then, we get

$$x_3 = \lambda x_1+(1-\lambda)x_4, \quad x_2 = \lambda x_4+(1-\lambda)x_1. \qquad (12)$$

If $a=0$, *i.e.* $X_s=[0,b]$ we have $x_1,x_2,x_3,x_4 \in X_s$ according to the assumption in Theorem 1. Since $d(x)$ is convex on X_s, with Eq. (12) we obtain

$$\begin{aligned} g(x_3) &\leq \lambda g(x_1)+(1-\lambda)g(x_4),\\ g(x_2) &\leq \lambda g(x_4)+(1-\lambda)g(x_1)\end{aligned} \qquad (13)$$

Summing the two equations in Eq. (13), we can get

$$g(x_2)+g(x_3) \leq g(x_1)+g(x_4)$$

Thus, $\theta(l_{i+1},l_j)-\theta(l_{i+1},l_{j+1})-\theta(l_i,l_j)+\theta(l_i,l_{j+1}) \geq 0$ is satisfied for any pair of label l_i, $l_j \in \mathcal{L}_s$.

If $a>0$ $(X_s=[a,b])$, we prove the theorem in three cases:
1) $i=j$; 2) $i>j+1$; 3) $i=j+1$.

1) When $i=j$, we have

$$\begin{aligned}&\theta(l_{i+1},l_j)-\theta(l_{i+1},l_{j+1})-\theta(l_i,l_j)+\theta(l_i,l_{j+1})\\ =&\theta(l_{i+1},l_i)-\theta(l_{i+1},l_{i+1})-\theta(l_i,l_i)+\theta(l_i,l_{i+1})\\ =&\theta(l_{i+1},l_i)-0-0+\theta(l_i,l_{i+1})\\ =&2\theta(l_{i+1},l_i) \geq 0\end{aligned}$$

2) When $i>j+1$, we have $x_1,x_2,x_3,x_4 \in X_s$ according to the assumption in Theorem 1. Since $d(x)$ is convex on X_s, with Eq. (12) we obtain

$$\begin{aligned} g(x_3) &\leq \lambda g(x_1)+(1-\lambda)g(x_4),\\ g(x_2) &\leq \lambda g(x_4)+(1-\lambda)g(x_1)\end{aligned} \qquad (14)$$

Summing the two equations in Eq. (14), we can get

$$g(x_2)+g(x_3) \leq g(x_1)+g(x_4)$$

Thus, $\theta(l_{i+1},l_j)-\theta(l_{i+1},l_{j+1})-\theta(l_i,l_j)+\theta(l_i,l_{j+1}) \geq 0$ is satisfied for any pair of label l_i, $l_j \in \mathcal{L}_s$ and $i>j+1$.

3) When $i=j+1$, we have

$$\begin{aligned} x_1 &= l_{j+2}-l_j, & x_2 &= l_{j+2}-l_{j+1},\\ x_3 &= l_{j+1}-l_j, & x_4 &= 0\end{aligned}$$

Thus, we have $x_1=x_2+x_3$, and $x_1,x_2,x_3 \in X_s$ but $x_4 \notin X_s$.

With Lemma 4, we have

$$\frac{g(x_1)-g(x_3)}{x_1-x_3} \geq \frac{g(x_2)-g(0)}{x_2}. \qquad (15)$$

Thus we can get

$$g(x_2)+g(x_3) \leq g(x_1)+g(x_4)$$

and $\theta(l_{i+1},l_j)-\theta(l_{i+1},l_{j+1})-\theta(l_i,l_j)+\theta(l_i,l_{j+1}) \geq 0$ is satisfied for any pair of labels l_i, $l_j \in \mathcal{L}_s$ and $i=j+1$.

Therefore, $\theta(l_{i+1},l_j)-\theta(l_{i+1},l_{j+1})-\theta(l_i,l_j)+\theta(l_i,l_{j+1}) \geq 0$ is satisfied for any pair of labels l_i, $l_j \in \mathcal{L}_s$. The proof is completed. □

Corollary 1 (Thereom 1)**.** *Assuming the interval $[a,b]$ is a candidate interval, then $\{\alpha, \alpha+x_1, \alpha+x_1+x_2, \cdots, \alpha+x_1+\cdots+x_m\} \subseteq \mathcal{L}$ is a submodular set, if $x_1,..,x_m \in [a,b]$ and $x_1+\cdots+x_m \leq b$.*

Proof. Let $\mathcal{L}_s=\{\alpha, \alpha+x_1, \alpha+x_1+x_2, \cdots, \alpha+x_1+\cdots+x_m\}$. We consider a pair of labels α_1 and α_2, which can be any pair of labels chosen in $\mathcal{L}_s$. According to the definition, there always exist p, q $(1 \leq p,q \leq m)$ such that

$$|\alpha_1-\alpha_2| = x_p+x_{p+1}+\cdots+x_q.$$

Since $x_i \in [a, b]$ for $\forall i \in [p, q]$, we have $|\alpha_1 - \alpha_2| \geq a$.

Since $x_1 + \cdots + x_m \leq b$, we have $x_p + x_{p+1} + \cdots + x_q \leq b$.

Thus, we have $|\alpha_1 - \alpha_2| \in [a, b]$ for any pair of labels $\alpha_1, \alpha_2 \in \mathcal{L}_s$

Thus, $\mathcal{L}_s$ is a submodular set according to Theorem 1.

$\square$