Robust 3D Hand Pose Estimation in Single Depth Images: from Single-View CNN to Multi-View CNNs Supplementary Material

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Derivation of the Optimal Solution

The optimization problem is formulated as follow:

$$\mathbf{\Phi}^* = \arg\min_{\mathbf{\Phi}} \sum_k \left(\boldsymbol{\phi}_k - \boldsymbol{\mu}_k\right)^T \mathbf{\Sigma}_k^{-1} \left(\boldsymbol{\phi}_k - \boldsymbol{\mu}_k\right)$$
$$s.t.\ \mathbf{\Phi} = \sum\nolimits_{m=1}^{M} \alpha_m \boldsymbol{e}_m + \boldsymbol{u} = \boldsymbol{E}\boldsymbol{\alpha} + \boldsymbol{u}$$

Let $R(\mathbf{\Phi}) = \sum\limits_k \left(\boldsymbol{\phi}_k - \boldsymbol{\mu}_k\right)^T \mathbf{\Sigma}_k^{-1} \left(\boldsymbol{\phi}_k - \boldsymbol{\mu}_k\right)$, which is in the quadratic form of the variable $\boldsymbol{\alpha}$, the optimal solution of $\boldsymbol{\alpha}$ can be obtained by setting the derivative of R with respect to $\boldsymbol{\alpha}$ to zero.

$$\frac{\partial R\left(\mathbf{\Phi}\right)}{\partial \boldsymbol{\alpha}} = \frac{\partial R\left(\mathbf{\Phi}\right)}{\partial \mathbf{\Phi}} \cdot \frac{\partial \mathbf{\Phi}\left(\boldsymbol{\alpha}\right)}{\partial \boldsymbol{\alpha}}$$

$$\because\ \frac{\partial R\left(\mathbf{\Phi}\right)}{\partial \boldsymbol{\phi}_k} = \frac{\partial}{\partial \boldsymbol{\phi}_k}\left[\left(\boldsymbol{\phi}_k - \boldsymbol{\mu}_k\right)^T \mathbf{\Sigma}_k^{-1} \left(\boldsymbol{\phi}_k - \boldsymbol{\mu}_k\right)\right] = 2\left(\boldsymbol{\phi}_k - \boldsymbol{\mu}_k\right)^T \mathbf{\Sigma}_k^{-1}$$

$$\therefore\ \frac{\partial R\left(\mathbf{\Phi}\right)}{\partial \mathbf{\Phi}} = 2\begin{bmatrix} \mathbf{\Sigma}_1^{-1}\left(\boldsymbol{\phi}_1 - \boldsymbol{\mu}_1\right) \\ \vdots \\ \mathbf{\Sigma}_k^{-1}\left(\boldsymbol{\phi}_k - \boldsymbol{\mu}_k\right) \\ \vdots \\ \mathbf{\Sigma}_K^{-1}\left(\boldsymbol{\phi}_K - \boldsymbol{\mu}_K\right) \end{bmatrix}^T$$

$$\because\ \frac{\partial \mathbf{\Phi}\left(\boldsymbol{\alpha}\right)}{\partial \boldsymbol{\alpha}} = \frac{\partial}{\partial \boldsymbol{\alpha}}\left(\boldsymbol{E}\boldsymbol{\alpha} + \boldsymbol{u}\right) = \boldsymbol{E} = \left[\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_M\right]$$

$$\therefore\ \frac{\partial R\left(\mathbf{\Phi}\right)}{\partial \boldsymbol{\alpha}} = 2\begin{bmatrix} \mathbf{\Sigma}_1^{-1}\left(\boldsymbol{\phi}_1 - \boldsymbol{\mu}_1\right) \\ \vdots \\ \mathbf{\Sigma}_k^{-1}\left(\boldsymbol{\phi}_k - \boldsymbol{\mu}_k\right) \\ \vdots \\ \mathbf{\Sigma}_K^{-1}\left(\boldsymbol{\phi}_K - \boldsymbol{\mu}_K\right) \end{bmatrix}^T \cdot \left[\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_M\right] = \mathbf{0}$$

Thus, we can get M linear equations for M unknown variables $\alpha_1, \alpha_2, \cdots, \alpha_M$:

$$\sum_{k=1}^{K}\left[\left(\sum_{j=1}^{M} \alpha_j \boldsymbol{e}_{jk}^T + \boldsymbol{u}_k^T - \boldsymbol{\mu}_k^T\right)\mathbf{\Sigma}_k^{-1}\boldsymbol{e}_{ik}\right] = 0,\quad i = 1,\ 2, \cdots,\ M$$

$$\therefore\ \sum_{j=1}^{M}\left[\left(\sum_{k=1}^{K} \boldsymbol{e}_{jk}^T \mathbf{\Sigma}_k^{-1} \boldsymbol{e}_{ik}\right)\alpha_j\right] = \sum_{k=1}^{K}\left(\boldsymbol{\mu}_k - \boldsymbol{u}_k\right)^T \mathbf{\Sigma}_k^{-1} \boldsymbol{e}_{ik}$$

Let $\mathbf{A}\boldsymbol{\alpha} = \boldsymbol{b}$, then:

$$\mathbf{A}_{ij} = \sum_{k=1}^{K} \boldsymbol{e}_{j,k}^T \mathbf{\Sigma}_k^{-1} \boldsymbol{e}_{i,k},\ \boldsymbol{b}_i = \sum_{k=1}^{K}\left(\boldsymbol{\mu}_k - \boldsymbol{u}_k\right)^T \mathbf{\Sigma}_k^{-1} \boldsymbol{e}_{i,k}\quad i,\ j = 1,\ 2, \cdots,\ M$$

At last, we can get the solution for the optimization problem: $\boldsymbol{\alpha}^* = \mathbf{A}^{-1}\boldsymbol{b}$.