Prior-Less Compressible Structure from Motion: Supplementary Material

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Recall the objective of Block Sparse Dictionary Learning (BSDL) is

$$\operatorname*{argmin}_{\mathbf{D},\boldsymbol{Z}} ||\boldsymbol{X} - \mathbf{D}\boldsymbol{Z}||_F^2 \quad \text{s.t.} ||\boldsymbol{Z}_i||_{0,\alpha} = K, \quad i = 1 : N/\beta, \tag{1}$$

where $\boldsymbol{Z}_i \in \mathbb{R}^{D\times\beta}$ is a submatrix of $\boldsymbol{Z}$, *i.e.* $\boldsymbol{Z} = [\boldsymbol{Z}_1, ..., \boldsymbol{Z}_{N/\beta}]$. Each $\boldsymbol{Z}_i$ is divided into $\boldsymbol{M}/\alpha$ blocks of size $\alpha \times \beta$ and $\|\boldsymbol{Z}_i\|_{0,\alpha}$ counts the number of blocks of which at least one element is non-zero. α and β need to be chosen such that D and M are perfectly divisible.

Definition 1. *If any valid solution $\{\hat{\mathbf{D}}, \hat{\boldsymbol{Z}}\}$ to the objective in Equation 1 is ambiguous only up to a $M \times M$ block permutation matrix $\boldsymbol{P}_\alpha$ and a block-diagonal invertible weighting matrix $\boldsymbol{\Lambda}_\alpha$ such that $\hat{\mathbf{D}} = \mathbf{D}\boldsymbol{P}_\alpha\boldsymbol{\Lambda}_\alpha$, and $\hat{\boldsymbol{Z}} = \boldsymbol{\Lambda}_\alpha^{-1}\boldsymbol{P}_\alpha^T\boldsymbol{Z}$, we say $\boldsymbol{X}$ has a unique BSDL.*

The block permutation matrix is actually defined as $\boldsymbol{P}_\alpha = \boldsymbol{P} \otimes \boldsymbol{I}_\alpha$ where $\boldsymbol{P}$ is an arbitrary $(M/\alpha) \times (M/\alpha)$ permutation matrix and $\boldsymbol{I}_\alpha$ is a $\alpha \times \alpha$ identity matrix. The block-diagonal invertible weighting matrix $\boldsymbol{\Lambda}_\alpha$ has a $\alpha \times \alpha$ block structure. We now ask the same question: what is the sufficient and necessary condition for the uniqueness of BSDL?

Theorem 1. *There exist $K\binom{M/\alpha}{K}^2$ K-block-sparse vectors $\boldsymbol{Z}_1, ..., \boldsymbol{Z}_{N/\beta}$,* i.e. *$N = \beta K\binom{M/\alpha}{K}^2$, such that the uniqueness of BSDL holds if and only if the matrix $\mathbf{D}$ satisfies the block spark condition:*

$$\mathbf{D}\boldsymbol{Z}_1 = \mathbf{D}\boldsymbol{Z}_2 \quad \textit{for K-block-sparse } \boldsymbol{Z}_1, \boldsymbol{Z}_2 \in \mathbb{R}^{M\times\beta} \Rightarrow \boldsymbol{Z}_1 = \boldsymbol{Z}_2. \tag{2}$$

Let's first prove Theorem 1 in the case when $\beta = 1$ and once it is proven, the general case $\beta > 1$ is simple to handle: We can split sparse causes $\boldsymbol{Z}^i$ into $[\mathbf{z}_1^i, \ldots, \mathbf{z}_\beta^i]$, where $\mathbf{z}_j^i \in \mathbb{R}^{D\times 1}$ and then $\mathbf{D}\boldsymbol{Z}^i = \mathbf{D}[\mathbf{z}_1^i, \ldots, \mathbf{z}_\beta^i] = \hat{\mathbf{D}}\hat{\boldsymbol{Z}}^i = \hat{\mathbf{D}}[\hat{\mathbf{z}}_1^i, \ldots, \hat{\mathbf{z}}_\beta^i]$ is equivalent to $\mathbf{D}\mathbf{z}_j^i = \hat{\mathbf{D}}\mathbf{z}_j^i$, which degenerates to the situation where $\beta = 1$.

A simple case when $K = 1$: To better understand Theorem 1 and prepare for the proof in full generality, let us start from a simple case when $K = 1$. Denote $\mathbf{e}_i^L$ as a L-dimensional column vector that has one in its i-th coordinate and zeros elsewhere. For convenience, let $L = M/\alpha$. Now let us produce M block vectors

$$\mathbf{z}_j^i = (\mathbf{e}_i^L \otimes \mathbf{e}_j^\alpha), \quad i = 1, \ldots, L, \quad j = 1, \ldots, \alpha, \tag{3}$$

which denotes that its j-th coordinate in i-th block is one and zeros elsewhere, and $L\binom{\alpha}{2}$ block vectors $\mathbf{z}_{jk}^i = \mathbf{z}_{jk}^i + \mathbf{z}_{jk}^i$, for any i and $j \neq k$.

Now we claim that the uniqueness of BSDL in this simple case can be achieved by these $M + L\binom{\alpha}{2}$ block vectors, which is less than $K\binom{M/\alpha}{K}^2$ assuming $M \gg \alpha$.

Proof. There exists a matrix $\hat{\mathbf{D}}$ and 1-block-sparse vector $\hat{\mathbf{z}}_j^i = (\mathbf{e}_{\pi(i,j)}^L \otimes \boldsymbol{I}_\alpha)\boldsymbol{\lambda}_{ij}$, for some mapping $\pi : \{1, ..., L\} \times \{1, ..., \alpha\} \rightarrow \{1, ..., L\}$ and $\boldsymbol{\lambda}_{ij} \in \mathbb{R}^\alpha$, such that

$$\mathbf{D}\mathbf{z}_j^i = \mathbf{D}(\mathbf{e}_i^L \otimes \mathbf{e}_j^\alpha) = \hat{\mathbf{D}}\hat{\mathbf{z}}_j^i = \hat{\mathbf{D}}(\mathbf{e}_{\pi(i,j)}^L \otimes I_\alpha)\boldsymbol{\lambda}_{ij}, \tag{4}$$

We claim that $\pi(i, j)$ is only dependent on i, not j. From Equation 4, we know that for any $j \neq k$, $\mathbf{D}\mathbf{z}_{jk}^i = \mathbf{D}(\mathbf{z}_j^i + \mathbf{z}_k^i) = \mathbf{D}\mathbf{z}_j^i + \mathbf{D}\mathbf{z}_k^i = \hat{\mathbf{D}}\hat{\mathbf{z}}_j^i + \hat{\mathbf{D}}\hat{\mathbf{z}}_k^i = \hat{\mathbf{D}}(\hat{\mathbf{z}}_j^i + \hat{\mathbf{z}}_k^i)$. Since $\mathbf{z}_{jk}^i$ is 1-block-sparse, this implies that $\hat{\mathbf{z}}_j^i + \hat{\mathbf{z}}_k^i$ should also be 1-block-sparse. Therefore $\pi(i, j) = \pi(i, k)$, that is, $\pi : \{1, ..., L\} \rightarrow \{1, ..., L\}$.

$$\mathbf{D}(\mathbf{e}_i^L \otimes \mathbf{e}_j^\alpha) = \hat{\mathbf{D}}(\mathbf{e}_{\pi(i)}^L \otimes \boldsymbol{I}_\alpha)\boldsymbol{\lambda}_{ij}. \tag{5}$$

Let us now prove that $\boldsymbol{\Lambda}_i = [\boldsymbol{\lambda}_{i1}, \ldots, \boldsymbol{\lambda}_{i\alpha}]$ is invertible. Let $\boldsymbol{Z}^i = [\mathbf{z}_1^i, \ldots, \mathbf{z}_\alpha^i]$ and $\hat{\boldsymbol{Z}}^i = [\hat{\mathbf{z}}_1^i, \ldots, \hat{\mathbf{z}}_\alpha^i]$. From Equation 5, it follows that $\mathbf{D}\boldsymbol{Z}^i = \mathbf{D}[\mathbf{z}_1^i, \ldots, \mathbf{z}_\alpha^i] = \mathbf{D}[(\mathbf{e}_i^L \otimes \mathbf{e}_1^\alpha), ..., (\mathbf{e}_i^L \otimes \mathbf{e}_\alpha^\alpha)] = \mathbf{D}(\mathbf{e}_i^L \otimes \boldsymbol{I}_\alpha)$, and $\mathbf{D}\boldsymbol{Z}^i = \hat{\mathbf{D}}\hat{\boldsymbol{Z}}^i = \hat{\mathbf{D}}(\mathbf{e}_{\pi(i)}^L \otimes \boldsymbol{I}_\alpha)\left[\boldsymbol{\lambda}_{i1}, ..., \boldsymbol{\lambda}_{i\alpha}\right] = \hat{\mathbf{D}}(\mathbf{e}_{\pi(i)}^L \otimes \boldsymbol{I}_\alpha)\boldsymbol{\Lambda}_i$. Therefore,

$$\mathbf{D}(\mathbf{e}_i^L \otimes \boldsymbol{I}_\alpha) = \hat{\mathbf{D}}(\mathbf{e}_{\pi(i)}^L \otimes \boldsymbol{I}_\alpha)\boldsymbol{\Lambda}_i. \tag{6}$$

Due to the fact that $\mathbf{D}$ satisfies the block spark condition, $\text{rank}(\mathbf{D}(\mathbf{e}_i^L \otimes \boldsymbol{I}_\alpha)) = \alpha$. From Equation 6, $\text{rank}(\hat{\mathbf{D}}(\mathbf{e}_{\pi(i)}^L \otimes \boldsymbol{I}_\alpha)\boldsymbol{\Lambda}_i) = \alpha$. We know that $\text{rank}(\boldsymbol{XY}) \leq \min(\text{rank}(\boldsymbol{X}), \text{rank}(\boldsymbol{Y}))$, for any matrix $\boldsymbol{X}, \boldsymbol{Y}$. So $\text{rank}(\boldsymbol{\Lambda}_i) \geq \alpha$. As $\boldsymbol{\Lambda}_i \in \mathbb{R}^{\alpha\times\alpha}$, $\text{rank}(\boldsymbol{\Lambda}_i) = \alpha$.

Now, let us show π is necessarily injective. Suppose $\pi(i) = \pi(j)$, with $i \neq j$, then from Equation 6, $\mathbf{D}(\mathbf{e}_i^L \otimes \boldsymbol{I}_\alpha) = \hat{\mathbf{D}}(\mathbf{e}_{\pi(i)}^L \otimes \boldsymbol{I}_\alpha)\boldsymbol{\Lambda}_i = \hat{\mathbf{D}}(\mathbf{e}_{\pi(j)}^L \otimes \boldsymbol{I}_\alpha)\boldsymbol{\Lambda}_j\boldsymbol{\Lambda}_j^{-1}\boldsymbol{\Lambda}_i = \mathbf{D}(\mathbf{e}_j^L \otimes \boldsymbol{I}_\alpha)\boldsymbol{\Lambda}_j^{-1}\boldsymbol{\Lambda}_i$. Since $\mathbf{D}$ satisfies the block spark condition, which implies $\mathbf{D}$ can never map two different 1-block-sparse vectors to the same measurement, this is possible only if $i = j$. Thus, π is injective.

Let $\boldsymbol{P}_\pi$ and $\mathbf{D}$ be generated by

$$\boldsymbol{P}_\pi = \begin{bmatrix} \mathbf{e}_{\pi(1)}^L & \cdots & \mathbf{e}_{\pi(K)}^L \end{bmatrix}, \boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{\Lambda}_L \end{bmatrix}. \tag{7}$$

Since π is injective, $\boldsymbol{P}_\pi$ is a permutation matrix. Let us stack Equation 6 from left-to-right on both sides, and it follows that on left sides,

$$[\mathbf{D}(\mathbf{e}_1^L \otimes \boldsymbol{I}_\alpha), \ldots, \mathbf{D}(\mathbf{e}_L^L \otimes \boldsymbol{I}_\alpha)] = \mathbf{D}, \tag{8}$$

and on right sides,

$$[\hat{\mathbf{D}}(\mathbf{e}_{\pi(1)}^L \otimes \boldsymbol{I}_\alpha)\boldsymbol{\Lambda}_1, \ldots, \hat{\mathbf{D}}(\mathbf{e}_{\pi(L)}^L \otimes \boldsymbol{I}_\alpha)\boldsymbol{\Lambda}_L] = \hat{\mathbf{D}}(\boldsymbol{P}_\pi \otimes \boldsymbol{I}_\alpha)\boldsymbol{\Lambda}. \tag{9}$$

Hence, we proved Theorem 1 for the simple case, where $K = 1$. □

Preparation: We use the same notation reported in [2]: Denote $[L]$ as the set $\{1, \ldots, L\}$ and $\binom{[L]}{K}$ as the K-element subset of $[L]$. Moreover, let the dictionary $\mathbf{D} = [\mathbf{D}_1, \ldots, \mathbf{D}_L]$ with $\mathbf{D}_i \in \mathbb{R}^{D\times\alpha}$, and denote $\text{span}\{\mathbf{D}_\mathcal{S}\}$ as a subspace expanded by $\mathbf{D}_i$, $i \in \mathcal{S}$.

To prove Theorem 1 in general situations, we offer a lemma at first.

Lemma 1. *Suppose that $\mathbf{D}$ satisfies the block spark condition and*

$$\kappa : \binom{[L]}{K} \to \binom{[L]}{K} \tag{10}$$

is a mapping with the following property: for all $\mathcal{S} \in \binom{[L]}{K}$,

$$\text{span}\{\mathbf{D}_\mathcal{S}\} = \text{span}\{\hat{\mathbf{D}}_{\kappa(\mathcal{S})}\}. \tag{11}$$

Then, there exist a permutation matrix $\boldsymbol{P}_\kappa \in \mathbb{R}^{L\times L}$ and an invertible block diagonal matrix $\boldsymbol{\Lambda} \in \mathbb{R}^{M\times M}$ such that $\mathbf{D} = \hat{\mathbf{D}}(\boldsymbol{P}_\kappa \otimes \boldsymbol{I}_\alpha)\boldsymbol{\Lambda}$.

Proof. Here we demonstrate, through induction, that if our $K = 1$ case holds, therefore, $K > 1$ case should also hold. First, let us show function κ is injective. Suppose that $\mathcal{S}, \mathcal{S}' \in \binom{[L]}{K}$ are different and $\kappa(\mathcal{S}) = \kappa(\mathcal{S}')$ holds. Then by Equation 11, $\text{span}\{\mathbf{D}_\mathcal{S}\} = \text{span}\{\hat{\mathbf{D}}_{\kappa(\mathcal{S})}\} = \text{span}\{\hat{\mathbf{D}}_{\kappa(\mathcal{S}')}\} = \text{span}\{\mathbf{D}_{\mathcal{S}'}\}$. As $\mathbf{D}$ satisfies the block spark condition, every $K+1$ block columns of $\mathbf{D}$ are linearly independent. From Lemma 2 (see below), it turns out that $\mathcal{S} = \mathcal{S}'$, which implies κ is injective.

Denote $\eta = \kappa^{-1}$ as the inverse of κ. Fix $\mathcal{S} = \{i_1, ..., i_{K-1}\} \in \binom{[L]}{K-1}$, and set $\mathcal{S}_1 = \mathcal{S} \cup \{p\}$ and $\mathcal{S}_2 = \mathcal{S} \cup \{q\}$ for some fixed $p, q \notin \mathcal{S}$ with $p \neq q$. Since $K < L$, $L - (K-1) > 1$, thus, it is always possible to find such p and q. From Equation 11, we obtain:

$$\text{span}\{\mathbf{D}_{\eta(\mathcal{S}_1)}\} = \text{span}\{\hat{\mathbf{D}}_{\mathcal{S}_1}\}, \tag{12}$$

$$\text{span}\{\mathbf{D}_{\eta(\mathcal{S}_2)}\} = \text{span}\{\hat{\mathbf{D}}_{\mathcal{S}_2}\}. \tag{13}$$

Let us intersect Equation 12 and Equation 13, and from Lemma 3 (see below) it follows that $\text{span}\{\hat{\mathbf{D}}_{\mathcal{S}_1}\} \cap \text{span}\{\hat{\mathbf{D}}_{\mathcal{S}_2}\} = \text{span}\{\mathbf{D}_{\eta(\mathcal{S}_1)\cap\eta(\mathcal{S}_2)}\}$. Since $\text{span}\{\hat{\mathbf{D}}_\mathcal{S}\} \subseteq \text{span}\{\hat{\mathbf{D}}_{\mathcal{S}_1}\} \cap \text{span}\{\hat{\mathbf{D}}_{\mathcal{S}_2}\}$, it follows that $\text{span}\{\hat{\mathbf{D}}_\mathcal{S}\} \subseteq \text{span}\{\mathbf{D}_{\eta(\mathcal{S}_1)\cap\eta(\mathcal{S}_2)}\}$. The number of the elements in $\eta(\mathcal{S}_1) \cap \eta(\mathcal{S}_2)$ is $K-1$, since $\eta(p) \neq \eta(q)$, with $p \neq q$, by injectivity of η. Moreover the number of the elements in $\mathcal{S}$ is also $K-1$, which implies that

$$\text{span}\{\hat{\mathbf{D}}_\mathcal{S}\} = \text{span}\{\mathbf{D}_{\eta(\mathcal{S}_1)\cap\eta(\mathcal{S}_2)}\}. \tag{14}$$

The association $\mathcal{S} \to \eta(\mathcal{S}_1) \cap \eta(\mathcal{S}_2)$ from Equation 14 defines a function $\sigma : \binom{[L]}{K-1} \to \binom{[L]}{K-1}$, with property that $\text{span}\{\hat{\mathbf{D}}_\mathcal{S}\} = \text{span}\{\mathbf{D}_{\sigma(\mathcal{S})}\}$.

Finally, let's show that σ is injective. Suppose $\mathcal{S}, \mathcal{S}' \in \binom{[L]}{K-1}$, and $\sigma(\mathcal{S}) = \sigma(\mathcal{S}')$, it follows that $\text{span}\{\hat{\mathbf{D}}_\mathcal{S}\} = \text{span}\{\mathbf{D}_{\sigma(\mathcal{S})}\} = \text{span}\{\mathbf{D}_{\sigma(\mathcal{S}')}\} = \text{span}\{\hat{\mathbf{D}}_{\mathcal{S}'}\}$. As every K block columns of $\mathbf{D}$ are linear independent, and κ is injective, every K block columns of $\hat{\mathbf{D}}$ are also linear independent. From Lemma 2, it follows that $\mathcal{S} = \mathcal{S}'$, which implies σ is injective. Hence, let $\xi = \sigma^{-1}$, with properties: for all $\mathcal{S} \in \binom{[L]}{K-1}$, $\text{span}\{\mathbf{D}_\mathcal{S}\} = \text{span}\{\hat{\mathbf{D}}_{\xi(\mathcal{S})}\}$. □

Lemma 2. *If any set of $K+1$ block columns of matrix $\mathbf{D} = [\mathbf{D}_1, \ldots, \mathbf{D}_L]$ are linear independent, then for $\mathcal{S}, \mathcal{S}' \in \binom{[L]}{K}$,*

$$\text{span}\{\mathbf{D}_\mathcal{S}\} = \text{span}\{\mathbf{D}_{\mathcal{S}'}\} \quad \Rightarrow \quad \mathcal{S} = \mathcal{S}'. \tag{15}$$

Proof. Suppose that $\mathcal{S} \neq \mathcal{S}' \in \binom{[L]}{K}$ satisfying $\text{span}\{\mathbf{D}_\mathcal{S}\} = \text{span}\{\mathbf{D}_{\mathcal{S}'}\}$. Then without loss of generality, there is an $i \in \mathcal{S}$ with $i \notin \mathcal{S}'$, but atoms $\mathbf{D}_i \in \text{span}\{\mathbf{D}_{\mathcal{S}'}\}$, which implies that the $K+1$ block columns indexed by $\mathcal{S}' \cup \{i\}$ are not linear independent, a contradiction to the assumption. □

Lemma 3. *If matrix $\mathbf{D}$ satisfies the block spark condition, then for $\mathcal{S}, \mathcal{S}' \in \binom{[L]}{K}$,*

$$\text{span}\{\mathbf{D}_{\mathcal{S}\cap\mathcal{S}'}\} = \text{span}\{\mathbf{D}_\mathcal{S}\} \cap \text{span}\{\mathbf{D}_{\mathcal{S}'}\}. \tag{16}$$

Proof. The inclusion "$\subseteq$" is trivial, so let us prove "$\supseteq$". Suppose a block vector $\mathbf{x} \in \text{span}\{\mathbf{D}_{\mathcal{S}}\} \cap \text{span}\{\mathbf{D}_{\mathcal{S}_2}\}$. Express $\mathbf{x}$ as a linear combination of K atoms of $\mathbf{D}$ indexed by $\mathcal{S}$ and, separately, as a combination of K atoms of $\mathbf{D}$ indexed by $\mathcal{S}'$. By the block spark condition, these linear combinations must be identical. In particular, $\mathbf{x}$ was expressed as a linear combination of atoms of $\mathbf{D}$ indexed by $\mathcal{S} \cap \mathcal{S}'$, and thus is in $\text{span}\{\mathbf{D}_{\mathcal{S}\cap\mathcal{S}'}\}$ □

Proof of Theorem 1 when $\beta = 1$: First, we produce a set of $N = K\binom{M/\alpha}{K}^2$ vectors $\boldsymbol{s}_i \in \mathbb{R}^{\alpha K}$ in general linear position (*i.e.* any subset of K of them are linearly independent). One possible strategy is to produce a "Vandermonde" matrix [3]. Next, we form K-block-sparse vectors $\mathbf{z}_1, ..., \mathbf{z}_N$ by taking $\boldsymbol{s}_i$ for the support value of $\mathbf{z}_i$ where each possible support set is represented $K\binom{M/\alpha}{K}$ times. We claim that these $\mathbf{z}_i$ always guarantee the uniqueness of BSDL.

Proof. Suppose there exists an alternate dictionary $\hat{\mathbf{D}}$ and a set of K-block-sparse vectors $\hat{\mathbf{z}}_1, ..., \hat{\mathbf{z}}_N$ such that $\mathbf{D}\mathbf{z}_i = \mathbf{x}_i = \hat{\mathbf{D}}\hat{\mathbf{z}}_i$. As there are $K\binom{M/\alpha}{K}$ $\mathbf{x}_i$ for each support indexed by $\mathcal{S}$, the "pigeon-hole principle"[1] implies that there are at least K vectors $\hat{\mathbf{z}}_{i_1}, ..., \hat{\mathbf{z}}_{i_K}$ using the same support $\mathcal{S}'$. Thus, $\text{span}\{\mathbf{x}_{i_1}, ..., \mathbf{x}_{i_K}\} \subseteq \text{span}\{\hat{\mathbf{D}}_{\mathcal{S}'}\}$. By the general linear position and the block spark condition, $\text{span}\{\mathbf{x}_{i_1}, ..., \mathbf{x}_{i_K}\} = \text{span}\{\mathbf{D}_{\mathcal{S}}\}$. Therefore $\text{span}\{\mathbf{D}_{\mathcal{S}}\} \subseteq \text{span}\{\hat{\mathbf{D}}_{\mathcal{S}'}\}$. As the dimension of $\text{span}\{\hat{\mathbf{D}}_{\mathcal{S}'}\}$ is less and equal to K, $\text{span}\{\mathbf{D}_{\mathcal{S}}\} = \text{span}\{\hat{\mathbf{D}}_{\mathcal{S}'}\}$.

By Lemma 1, Theorem 1 is proved. □

Discussion: A lower $N = (K+1)\binom{M}{K}$ is offered by Hillar *et al.*'s probabilistic theorems in [2] saying that if $K+1$ K-sparse vector $\mathbf{z}_i$ are randomly drawn from each support set, and $\mathbf{D}$ satisfies the spark condition, then $\boldsymbol{X}$ has a unique SDL with probability one. We hypothesize that a lower $N = (K+1)\binom{M/\alpha}{K}$ is also enough for the uniqueness of BSDL to hold with probability one, which will be a focus of future work.

References

[1] R. A. Brualdi. *Introductory combinatorics*. New York, 1992. 3

[2] C. Hillar and F. T. Sommer. When can dictionary learning uniquely recover sparse data from subsamples? In *IEEE Transactions on Information Theory*, 2015. 2, 3

[3] L. R. Turner. Inverse of the vandermonde matrix with applications. 1966. 3

[1] The pigeon-hole principle states that if n items are put into m containers, with $n > m$, then at least one container must contain more than one item [1].