Efficient Intersection of Three Quadrics and Applications in Computer Vision
Appendix

Zuzana Kukelova[1] Jan Heller[2] Andrew Fitzgibbon[1]

[1]Microsoft Research Ltd,
21 Station Road,
Cambridge CB1 2FB, UK
{zukuke,awf}@microsoft.com

[2]Czech Technical University
in Prague, 166 27 Praha 6,
Technická 2, Czech Republic
{hellej1}@cmp.felk.cvut.cz

The general E3Q3 solver introduced in the main paper is able to quickly and reliably solve non-degenerate configurations of the 3Q3 problem. In this Appendix, §A.1, we present the complete E3Q3 solver, *i.e.*, the general E3Q3 solver supplemented with seven particular solvers covering seven distinct degenerate configurations. Next, in §A.2 we discuss the behavior of the E3Q3 solver in situation where the symmetry or solutions with multiplicities are involved. Further, in §A.3 we present a novel 3Q3 formulation of the hand-eye calibration problem with known translation. Finally, in §A.4, we provide the full parametrization of the projection matrix $\mathtt{P}$ for the new 3Q3 formulation of problem P4Pf from the main paper.

A.1. E3Q3: The complete 3Q3 solver

Let us here briefly review the 3Q3 problem formulation.

Let x, y, z be the problem unknowns, c_{ij}, $i = 1,2,3$, $j = 1,\dots,10$ the problem coefficients and (with $\mathbf{c}_i = [c_{i1}, c_{i2}, \dots, c_{i10}]$),

$$q_i = \mathbf{c}_i \cdot [x^2, y^2, z^2, xy, xz, yz, x, y, z, 1] \tag{1}$$

the three polynomials of degree 2. The problem of 3Q3 is to find the intersections of q_i, *i.e.*, to solve the system

$$q_i = 0,\ \ i = 1,2,3. \tag{2}$$

To solve Eqs. 2, we will start by 'hiding' the unknown x in the coefficient field, *i.e.*, by considering x to be a coefficient for a moment. This leaves us with six monomials $[y^2, z^2, yz, y, z, 1]$ in unknowns y, z in every equation. By splitting these monomials into two sets $\{y^2, z^2, yz\}$, $\{y, z, 1\}$ and by rearranging them into the left- and right-hand sides, Eqs. 2 can be rewritten as the following matrix equation:

$$\mathtt{A}\begin{bmatrix} y^2 \\ z^2 \\ yz \end{bmatrix} = \begin{bmatrix} p_{11}^{[1]}(x) & p_{12}^{[1]}(x) & p_{13}^{[2]}(x) \\ p_{21}^{[1]}(x) & p_{22}^{[1]}(x) & p_{23}^{[2]}(x) \\ p_{31}^{[1]}(x) & p_{32}^{[1]}(x) & p_{33}^{[2]}(x) \end{bmatrix}\begin{bmatrix} y \\ z \\ 1 \end{bmatrix}, \tag{3}$$

where $\mathtt{A}$ is a coefficient matrix

$$\mathtt{A} = \begin{bmatrix} c_{12} & c_{13} & c_{16} \\ c_{22} & c_{23} & c_{26} \\ c_{32} & c_{33} & c_{36} \end{bmatrix}, \tag{4}$$

and $p_{11}(x), \dots, p_{33}(x)$ are polynomials in x. where the upper index $[\cdot]$ denotes the maximum possible degree of the respective polynomial $p_{ij}(x)$.

By using G-J elimination and, if necessary, by interchanging of the matrix rows, the structure of the matrix $\mathtt{A}$ can be reduced into one of the following configurations:

$$\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&\bullet\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\0&0&1\\0&0&0 \end{bmatrix},$$
$$\begin{bmatrix} 1&\bullet&\bullet\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 1&\bullet&0\\0&0&1\\0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&\bullet\\0&1&\bullet\\0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}, \tag{5}$$

where $\bullet$ stands for a nonzero coefficient from $\mathbb{R}$. All other configurations can be transformed into one of these 8 configurations either by interchanging variables y and z or by assuming $\bullet$ equal to 0. In the next, we will denote these configurations by Roman numerals as Cases I, II, ..., VIII, and for each case we will derive a particular solver. Note that Case VIII corresponds to the non-degenerate configuration solved in §3 of the main paper.

For the degenerate configurations I through VII, we will follow the idea of the solution for the general Case VIII.

This means that we will derive three equations that are linear in y and z. Such equations can be written in the form

$$\mathtt{M}(x)\begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathbf{0}, \tag{6}$$

where $\mathtt{M}(x)$ is a polynomial matrix which entries are polynomials in x. As we know from elementary linear algebra, matrix Eq. 6 has a non-trivial solution if and only if the determinant of its matrix $\mathtt{M}(x)$ is zero. The solutions to the unknown x can then be obtained by finding the roots of the single variable polynomial $\det(\mathtt{M}(x)) = 0$. Solutions to the unknowns y and z can be obtained from $\mathtt{M}(x)$ after substituting the particular solutions for x into this matrix from Eq. 6 and solving the resulting system of two linear equations.

A.1.1. Case I

In the configuration of Case I, the matrix $\mathtt{A}$ takes the form

$$\mathtt{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

and Eq. 3 can be rewritten as

$$\begin{bmatrix} p_{11}^{[1]}(x) & p_{12}^{[1]}(x) & p_{13}^{[2]}(x) \\ p_{21}^{[1]}(x) & p_{22}^{[1]}(x) & p_{23}^{[2]}(x) \\ p_{31}^{[1]}(x) & p_{32}^{[1]}(x) & p_{33}^{[2]}(x) \end{bmatrix} \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathtt{M}(x)\begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathbf{0}, \tag{8}$$

It can be easily inferred from the degrees of $p_{ij}(x)$ that the determinant of the 3×3 polynomial matrix $\mathtt{M}(x)$ is a up to degree 4 polynomial in x. The solutions to the unknown x from the original Eq. 2 can now be obtained by finding the roots of $\det(\mathtt{M}(x))$ using Sturm sequences [7] in some feasible interval.

The particular solver for Case I needs to perform 67 additions and 144 multiplications.

A.1.2. Case II

In the configuration of Case II, the matrix $\mathtt{A}$ takes the form

$$\mathtt{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{9}$$

and Eq. 3 can be rewritten as

$$p_{11}^{\prime[1]}(x)y + p_{12}^{\prime[1]}(x)z + p_{13}^{\prime[2]}(x) = yz, \tag{10}$$
$$p_{21}^{\prime[1]}(x)y + p_{22}^{\prime[1]}(x)z + p_{23}^{\prime[2]}(x) = 0, \tag{11}$$
$$p_{31}^{\prime[1]}(x)y + p_{32}^{\prime[1]}(x)z + p_{33}^{\prime[2]}(x) = 0, \tag{12}$$

where $p_{11}^{\prime[1]}(x), \ldots, p_{33}^{\prime[2]}(x)$ are linear combinations of polynomials $p_{11}^{[1]}(x), \ldots, p_{33}^{[2]}(x)$.

Our goal is to obtain three polynomial equations that are linear in y and z. Eqs. 11 and 12 already have the required form. Eq. 10 contains monomial yz that needs to be eliminated. Let us rewrite Eqs. 11 and 12 as

$$p_{21}^{\prime[1]}(x)y = -p_{22}^{\prime[1]}(x)z - p_{23}^{\prime[2]}(x), \tag{13}$$
$$p_{32}^{\prime[1]}(x)z = -p_{31}^{\prime[1]}(x)y - p_{33}^{\prime[2]}(x). \tag{14}$$

By multiplying Eqs. 13 and 14 and substituting the expression for yz from Eq. 10 we obtain

$$\begin{aligned} & p_{21}^{\prime[1]}(x)p_{32}^{\prime[1]}(x)yz \\ &= (p_{22}^{\prime[1]}(x)z + p_{23}^{\prime[2]}(x))(p_{31}^{\prime[1]}(x)y + p_{33}^{\prime[2]}(x)) \\ &= p_{22}^{\prime[1]}(x)p_{31}^{\prime[1]}(x)yz + p_{23}^{\prime[2]}(x)p_{31}^{\prime[1]}(x)y \\ &\quad + p_{22}^{\prime[1]}(x)p_{33}^{\prime[2]}(x)z + p_{23}^{\prime[2]}(x)p_{33}^{\prime[2]}(x) \\ &= p_{11}^{\prime[1]}(x)p_{22}^{\prime[1]}(x)p_{31}^{\prime[1]}(x)y + p_{12}^{\prime[1]}p_{22}^{\prime[1]}(x)p_{31}^{\prime[1]}(x)z \\ &\quad + p_{13}^{\prime[2]}(x)p_{22}^{\prime[1]}(x)p_{31}^{\prime[1]}(x) + p_{23}^{\prime[2]}(x)p_{31}^{\prime[1]}(x)y \\ &\quad + p_{22}^{\prime[1]}(x)p_{33}^{\prime[2]}(x)z + p_{23}^{\prime[2]}p_{33}^{\prime[2]}. \end{aligned} \tag{15}$$

Now, Let us multiply the Eq. 10 with $p_{21}^{\prime[1]}p_{32}^{\prime[1]}$. This results in

$$\begin{aligned} & p_{21}^{\prime[1]}(x)p_{32}^{\prime[1]}(x)yz \\ &= p_{21}^{\prime[1]}(x)p_{32}^{\prime[1]}(x)\left(p_{11}^{\prime[1]}(x)y + p_{12}^{\prime[1]}(x)z + p_{13}^{\prime[2]}(x)\right). \end{aligned} \tag{16}$$

By combining the right-hand sides of Eqs. 15 and 16 we obtain an equation of the form

$$s_{11}^{[3]}(x)y + s_{12}^{[3]}(x)z + s_{13}^{[4]}(x) = 0, \tag{17}$$

where $s_{11}^{[3]}(x), s_{12}^{[3]}(x), \ldots, s_{13}^{[4]}(x)$ are polynomials in x

Finally, by stacking Eqs. 11, 12 and 17 into a matrix form we get

$$\begin{bmatrix} p_{21}^{[1]}(x) & p_{22}^{[1]}(x) & p_{23}^{[2]}(x) \\ p_{31}^{[2]}(x) & p_{32}^{[1]}(x) & p_{33}^{[2]}(x) \\ s_{11}^{[3]}(x) & s_{12}^{[3]}(x) & s_{13}^{[4]}(x) \end{bmatrix} \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathtt{M}(x)\begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathbf{0}. \tag{18}$$

After finding the roots of the up to degree 6 polynomial $\det(\mathtt{M}(x))$ for $\mathtt{M}(x)$ from Eq. 18, we obtain the solutions to the unknown x.

The particular solver for Case II needs to perform 193 additions and 391 multiplications.

A.1.3. Case III

In the configuration of Case III, the matrix $\mathtt{A}$ takes the form

$$\mathtt{A} = \begin{bmatrix} 0 & 1 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{19}$$

and Eq. 3 can be rewritten as

$$p'^{[1]}_{11}(x)y + p'^{[1]}_{12}(x)z + p'^{[2]}_{13}(x) = z^2 + \alpha yz, \quad (20)$$
$$p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x) = 0, \quad (21)$$
$$p'^{[1]}_{31}(x)y + p'^{[1]}_{32}(x)z + p'^{[2]}_{33}(x) = 0, \quad (22)$$

where $p'^{[1]}_{11}(x), \ldots, p'^{[2]}_{33}(x)$ are linear combinations of polynomials $p^{[1]}_{11}(x), \ldots, p^{[2]}_{33}(x)$ and $\alpha \in \mathbb{R}$ is a constant.

Again, Eqs. 21 and 22 are already linear in y and z. Eq. 20 contains two second degree monomials yz and z^2 that need to be eliminated.

Let us start by combining Eqs. 21 and 22 to eliminate either y or z from these equations. This leads to the following two equations

$$\begin{aligned}&(p'^{[1]}_{31}(x)p'^{[1]}_{22}(x) - p'^{[1]}_{21}(x)p'^{[1]}_{32}(x))y \\ &\quad + (p'^{[2]}_{33}(x)p'^{[1]}_{22}(x) - p'^{[2]}_{23}(x)p'^{[1]}_{32}(x)) = 0, \quad (23)\\ &(p'^{[1]}_{31}(x)p'^{[1]}_{22}(x) - p'^{[1]}_{21}(x)p'^{[1]}_{32}(x))z \\ &\quad + (p'^{[1]}_{31}(x)p'^{[2]}_{23}(x) - p'^{[1]}_{21}(x)p'^{[2]}_{33}(x)) = 0. \quad (24)\end{aligned}$$

Now, let us introduce three new polynomials to simplify the notation:

$$s^{[2]}_1(x) = p'^{[1]}_{31}(x)p'^{[1]}_{22}(x) - p'^{[1]}_{21}(x)p'^{[1]}_{32}(x), \quad (25)$$
$$s^{[3]}_2(x) = p'^{[2]}_{33}(x)p'^{[1]}_{22}(x) - p'^{[2]}_{23}(x)p'^{[1]}_{32}(x), \quad (26)$$
$$s^{[3]}_3(x) = p'^{[1]}_{31}(x)p'^{[2]}_{23}(x) - p'^{[1]}_{21}(x)p'^{[2]}_{33}(x). \quad (27)$$

By multiplying Eqs. 23 and 24 by z we obtain

$$s^{[2]}_1(x)yz + s^{[3]}_2(x)z = 0, \quad (28)$$
$$s^{[2]}_1(x)z^2 + s^{[3]}_3(x)z = 0. \quad (29)$$

Further, let us multiply Eq. 20 by $s^{[2]}_1(x)$ and substitute the expressions for $s^{[2]}_1(x)yz$ and $s^{[2]}_1(x)z^2$ from Eqs. 28 and 29 to obtain an equation that is linear in y and z:

$$\begin{aligned}&- s^{[3]}_3(x)z - \alpha\, s^{[3]}_2(x)z \\ &\quad = s^{[2]}_1(x)(p'^{[1]}_{11}(x)y + p'^{[1]}_{12}(x)z + p'^{[2]}_{13}(x)). \quad (30)\end{aligned}$$

Finally, by denoting the polynomial coefficients of Eq. 30 as $s^{[3]}_{11}(x), s^{[3]}_{12}(x)$, and $s^{[4]}_{13}(x)$ we can stack Eqs. 21, 22, and 30 into a following matrix form

$$\begin{bmatrix} p^{[1]}_{21}(x) & p^{[1]}_{22}(x) & p^{[2]}_{23}(x) \\ p^{[2]}_{31}(x) & p^{[1]}_{32}(x) & p^{[2]}_{33}(x) \\ s^{[3]}_{11}(x) & s^{[3]}_{12}(x) & s^{[4]}_{13}(x) \end{bmatrix} \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathtt{M}(x) \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathbf{0}. \quad (31)$$

Again, $\det(\mathtt{M}(x))$ is a up to degree 6 polynomial which roots are the solutions for the unknown x.

The particular solver for Case III needs to perform 206 additions and 377 multiplications.

A.1.4. Case IV

In the configuration of Case IV, the matrix $\mathtt{A}$ takes the form

$$\mathtt{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (32)$$

and Eq. 3 can be rewritten as

$$p'^{[1]}_{11}(x)y + p'^{[1]}_{12}(x)z + p'^{[2]}_{13}(x) = z^2, \quad (33)$$
$$p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x) = yz, \quad (34)$$
$$p'^{[1]}_{31}(x)y + p'^{[1]}_{32}(x)z + p'^{[2]}_{33}(x) = 0, \quad (35)$$

where $p'^{[1]}_{11}(x), \ldots, p'^{[2]}_{33}(x)$ are linear combinations of polynomials $p^{[1]}_{11}(x), \ldots, p^{[2]}_{33}(x)$.

In this case, only Eq. 35 is linear in y and z. This means that we need to eliminate the degree 2 monomials z^2 and yz from Eqs. 33 and 34 and use these equations to construct two new polynomial equations that are linear in y and z.

To construct the first equation, let us multiply Eq. 35 by z and substitute the left hand sides of Eqs. 33 and 34 for monomials z^2 and yz, respectively. This leads to the following equation:

$$\begin{aligned}&p'^{[1]}_{31}(x)yz + p'^{[1]}_{32}(x)z^2 + p'^{[2]}_{33}(x)z \\ &\quad = p'^{[1]}_{31}(x)(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x)) \quad (36)\\ &\qquad + p'^{[1]}_{32}(x)(p'^{[1]}_{11}(x)y + p'^{[1]}_{12}(x)z + p'^{[2]}_{13}(x)) \\ &\qquad\qquad + p'^{[2]}_{33}(x)z = 0.\end{aligned}$$

Eq. 36 is linear in y and z. Let us denote the polynomial coefficients of Eq. 36 w.r.t. the monomials y, z, and 1 as $s^{[2]}_{11}(x), s^{[2]}_{12}(x)$, and $s^{[3]}_{13}(x)$.

To obtain the second equation, we can multiply the Eq. 35 by y to obtain an equation for $p'^{[1]}_{31}(x)y^2$:

$$\begin{aligned}&p'^{[1]}_{31}(x)y^2 \\ &\quad = -p'^{[1]}_{32}(x)yz - p'^{[2]}_{33}(x)y \quad (37)\\ &\quad = -p'^{[1]}_{32}(x)(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x)) \\ &\qquad\qquad - p'^{[2]}_{33}(x)y = 0.\end{aligned}$$

Now let us state a trivial identity

$$p'^{[1]}_{31}(x)yzz = p'^{[1]}_{31}(x)z^2y. \quad (38)$$

Let us recursively substitute the left-hand sides of Eqs. 33 and 34 for the monomials z^2 and yz, respectively, into the left-hand side of Eq. 38:

$$\begin{aligned}&p'^{[1]}_{31}(x)yzz \\ &\quad = p'^{[1]}_{31}(x)(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x))z \quad (39)\\ &\quad = p'^{[1]}_{31}(x)p'^{[1]}_{21}(x)(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x)) \\ &\qquad + p'^{[1]}_{31}(x)p'^{[1]}_{22}(x)(p'^{[1]}_{11}(x)y + p'^{[1]}_{12}(x)z + p'^{[2]}_{13}(x)) \\ &\qquad + p'^{[1]}_{31}(x)p'^{[2]}_{23}(x)z.\end{aligned}$$

Analogously, let us substitute the left-hand sides of Eqs. 33 and 34 into the right-hand side of Eq. 38. Further, let us substitute the expression from Eq. 37 for the polynomial $p_{31}'^{[1]}(x)y^2$. After recursively performing these substitutions, the right-hand side of Eq. 38 will read

$$
\begin{aligned}
& p_{31}'^{[1]}(x)z^2y \\
& \quad = p_{31}'^{[1]}(x)(p_{11}'^{[1]}(x)y + p_{12}'^{[1]}(x)z + p_{13}'^{[2]}(x))y \qquad (40)\\
& \quad = p_{11}'^{[1]}(x)\left(-p_{32}'^{[1]}(x)(p_{21}'^{[1]}(x)y + p_{22}'^{[1]}(x)z + p_{23}'^{[2]}(x)) + \right. \\
& \qquad \left. -p_{33}'^{[2]}(x)y\right) + p_{31}'^{[1]}(x)p_{12}'^{[1]}(x)\left(p_{21}'^{[1]}(x)y + p_{22}'^{[1]}(x)z\right. \\
& \qquad \left. + p_{23}'^{[2]}(x)\right) + p_{31}'^{[1]}(x)p_{13}'^{[2]}(x)y.
\end{aligned}
$$

By combining the right-hand sides of Eqs. 39 and 40 and by denoting the polynomial coefficients of the final polynomial as $s_{21}^{[3]}(x), s_{22}^{[3]}(x)$, and $s_{23}^{[4]}(x)$ we end up with the second new polynomial equation linear in y and z:

$$
s_{21}'^{[3]}(x)y + s_{22}'^{[3]}(x)z + s_{23}'^{[4]}(x) = 0. \qquad (41)
$$

Finally, we can stack Eqs. 35, 36 and 41 into a matrix form

$$
\begin{bmatrix} p_{31}^{[1]}(x) & p_{32}^{[1]}(x) & p_{33}^{[2]}(x) \\ s_{11}^{[2]}(x) & s_{12}^{[2]}(x) & s_{13}^{[3]}(x) \\ s_{21}^{[3]}(x) & s_{22}^{[3]}(x) & s_{23}^{[4]}(x) \end{bmatrix} \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathtt{M}(x) \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathbf{0}, \qquad (42)
$$

This time, $\det(\mathtt{M}(x))$ leads to a up to degree 7 polynomial. By finding its roots we find the solutions for the unknown x.

The particular solver for Case IV needs to perform 308 additions and 596 multiplications.

A.1.5. Case V

In the configuration of Case V, the matrix $\mathtt{A}$ takes the form

$$
\mathtt{A} = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (43)
$$

and Eq. 3 can be rewritten as

$$
\begin{aligned}
& p_{11}'^{[1]}(x)y + p_{12}'^{[1]}(x)z + p_{13}'^{[2]}(x) = y^2 + \alpha z^2 + \beta yz, \qquad (44)\\
& p_{21}'^{[1]}(x)y + p_{22}'^{[1]}(x)z + p_{23}'^{[2]}(x) = 0, \qquad (45)\\
& p_{31}'^{[1]}(x)y + p_{32}'^{[1]}(x)z + p_{33}'^{[2]}(x) = 0, \qquad (46)
\end{aligned}
$$

where $p_{11}'^{[1]}(x), \dots, p_{33}'^{[2]}(x)$ are linear combinations of polynomials $p_{11}^{[1]}(x), \dots, p_{33}^{[2]}(x)$ and $\alpha, \beta \in \mathbb{R}$ are constants.

Again, we already have two equations that are linear in y and z, Eqs. 45 and 46. In the next, we will use Eq. 44 to construct the third equation.

Analogous to Case III, Eqs. 45 and 46 can be combined to eliminate either y or z, resulting in the following two equations:

$$
\begin{aligned}
& (p_{31}'^{[1]}(x)p_{22}'^{[1]}(x) - p_{21}'^{[1]}(x)p_{32}'^{[1]}(x))y \\
& + (p_{33}'^{[2]}(x)p_{22}'^{[1]}(x) - p_{23}'^{[2]}(x)p_{32}'^{[1]}(x)) = 0, \qquad (47)\\
& (p_{31}'^{[1]}(x)p_{22}'^{[1]}(x) - p_{21}'^{[1]}(x)p_{32}'^{[1]}(x))z \\
& + (p_{31}'^{[1]}(x)p_{23}'^{[2]}(x) - p_{21}'^{[1]}(x)p_{33}'^{[2]}(x)) = 0. \qquad (48)
\end{aligned}
$$

Now, let us introduce three new polynomials to simplify the notation:

$$
\begin{aligned}
s_1^{[2]}(x) &= p_{31}'^{[1]}(x)p_{22}'^{[1]}(x) - p_{21}'^{[1]}(x)p_{32}'^{[1]}(x), \qquad (49)\\
s_2^{[3]}(x) &= p_{33}'^{[2]}(x)p_{22}'^{[1]}(x) - p_{23}'^{[2]}(x)p_{32}'^{[1]}(x), \qquad (50)\\
s_3^{[3]}(x) &= p_{31}'^{[1]}(x)p_{23}'^{[2]}(x) - p_{21}'^{[1]}(x)p_{33}'^{[2]}(x). \qquad (51)
\end{aligned}
$$

By multiplying Eq. 47 by y and by z, respectively, and Eq. 48 by z, we obtain

$$
\begin{aligned}
s_1^{[2]}(x)y^2 + s_2^{[3]}(x)y &= 0, \qquad (52)\\
s_1^{[2]}(x)yz + s_2^{[3]}(x)z &= 0, \qquad (53)\\
s_1^{[2]}(x)z^2 + s_3^{[3]}(x)z &= 0. \qquad (54)
\end{aligned}
$$

Further, let us multiply Eq. 44 by the polynomial $s_1^{[2]}(x)$ and substitute the expressions for the polynomials $s_1^{[2]}(x)y^2$, $s_1^{[2]}(x)yz$, and $s_1^{[2]}(x)z^2$ from Eqs. 52, 53, and 53, respectively, to obtain the third polynomial equation that is linear in y and z:

$$
\begin{aligned}
& - s_2^{[3]}(x)y - \alpha s_3^{[3]}(x)z - \beta s_2^{[3]}(x)z \\
& \qquad = s_1^{[2]}(x)(p_{11}'^{[1]}(x)y + p_{12}'^{[1]}(x)z + p_{13}'^{[2]}(x)). \qquad (55)
\end{aligned}
$$

Finally, by denoting the polynomial coefficients of Eq. 55 as $s_{11}^{[3]}(x), s_{12}^{[3]}(x)$, and $s_{13}^{[4]}(x)$, we can stack Eqs. 45, 46, and 55 into a following matrix form

$$
\begin{bmatrix} p_{21}^{[1]}(x) & p_{22}^{[1]}(x) & p_{23}^{[2]}(x) \\ p_{31}^{[2]}(x) & p_{32}^{[1]}(x) & p_{33}^{[2]}(x) \\ s_{11}^{[3]}(x) & s_{12}^{[3]}(x) & s_{13}^{[4]}(x) \end{bmatrix} \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathtt{M}(x) \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathbf{0}, \qquad (56)
$$

The solutions to the unknown x can now be obtained by finding the roots of the up to degree 6 polynomial $\det(\mathtt{M}(x))$ from Eq. 56.

The particular solver for Case V needs to perform 213 additions and 393 multiplications.

A.1.6. Case VI

In the configuration of Case VI, the matrix $\mathtt{A}$ takes the form

$$
\mathtt{A} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (57)
$$

and Eqs. 3 can be rewritten as

$$p'^{[1]}_{11}(x)y + p'^{[1]}_{12}(x)z + p'^{[2]}_{13}(x) = y^2 + \alpha z^2, \quad (58)$$
$$p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x) = yz, \quad (59)$$
$$p'^{[1]}_{31}(x)y + p'^{[1]}_{32}(x)z + p'^{[2]}_{33}(x) = 0, \quad (60)$$

where $p'^{[1]}_{11}(x), \ldots, p'^{[2]}_{33}(x)$ are linear combinations of polynomials $p^{[1]}_{11}(x), \ldots, p^{[2]}_{33}(x)$ and $\alpha \in \mathbb{R}$ is a constant.

In this case, only Eq. 60 is linear in y and z, *i.e.*, we need to eliminate the degree 2 monomials z^2 and yz from Eqs. 58 and 59 and use them to construct two new polynomial equations, both linear in y and z.

To obtain the first new equation, let us multiply Eq. 60 by y and by z to obtain the expressions for $p'^{[1]}_{31}(x)y^2$ and $p'^{[1]}_{32}(x)z^2$, respectively,

$$\begin{aligned} &p'^{[1]}_{31}(x)y^2 \\ &\quad = -p'^{[1]}_{32}(x)yz - p'^{[2]}_{33}(x)y \qquad (61)\\ &\quad = -p'^{[1]}_{32}(x)(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x)) \\ &\qquad - p'^{[2]}_{33}(x)y, \\ &p'^{[1]}_{32}(x)z^2 \\ &\quad = -p'^{[1]}_{31}(x)yz - p'^{[2]}_{33}(x)z \qquad (62)\\ &\quad = -p'^{[1]}_{31}(x)(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x)) \\ &\qquad - p'^{[2]}_{33}(x)z. \end{aligned}$$

Now, let us multiply Eq. 58 by the polynomial $p'^{[1]}_{31}(x)\, p'^{[1]}_{32}(x)$ and substitute the expressions for $p'^{[1]}_{31}(x)y^2$ and $p'^{[1]}_{32}(x)z^2$ from Eqs. 61 and 62 into this equation:

$$\begin{aligned} &p'^{[1]}_{31}(x)p'^{[1]}_{32}(x)(p'^{[1]}_{11}(x)y + p'^{[1]}_{12}(x)z + p'^{[2]}_{13}(x)) \qquad (63)\\ &\quad = p'^{[1]}_{32}(x)\left(-p'^{[1]}_{32}(x)(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x))\right. \\ &\qquad \left. - p'^{[2]}_{33}(x)y\right) + \alpha p'^{[1]}_{31}(x)\left(-p'^{[1]}_{31}(x)(p'^{[1]}_{21}(x)y\right. \\ &\qquad \left. + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x)) - p'^{[2]}_{33}(x)z\right). \end{aligned}$$

Eq. 63 is linear in y and z. Let us denote the polynomial coefficients of Eq. 63 w.r.t. the monomials y, z, and 1 as $s^{[3]}_{11}(x)$, $s^{[3]}_{12}(x)$, and $s^{[4]}_{13}(x)$.

To obtain the second new equation, let us first derive three equations that, after 'hiding' x in the coefficient field, contain only monomials z^2, z, y, and 1. Eq. 62 already has this special form. The second equation that contains only this set of monomials can be obtained by substituting Eq. 61 to Eq. 58 multiplied by $p'^{[1]}_{31}(x)$. This leads to the following equation:

$$\begin{aligned} &\alpha p'^{[1]}_{31}(x)z^2 \\ &\quad = p'^{[1]}_{32}(x)(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x)) \qquad (64)\\ &\qquad + p'^{[2]}_{33}(x)y + p'^{[1]}_{31}(p'^{[1]}_{11}(x)y + p'^{[1]}_{12}(x)z + p'^{[2]}_{13}(x)). \end{aligned}$$

The third equation containing monomials z^2, z, y, and 1 only can be obtained by expressing the polynomial $p'^{[1]}_{32}(x)y^2z$ in two different ways. First, let us multiply Eq. 59 by $p'^{[1]}_{32}(x)y$ and substitute the expressions for $p'^{[1]}_{32}(x)z^2$, y^2, and yz from Eqs. 62, 58, and 59, respectively:

$$\begin{aligned} &p'^{[1]}_{32}(x)y^2z \\ &\quad = p'^{[1]}_{32}(x)(p'^{[1]}_{21}(x)y^2 + p'^{[1]}_{22}(x)yz + p'^{[2]}_{23}(x)y) \\ &\quad = p'^{[1]}_{21}(x)p'^{[1]}_{32}(x)\left(p'^{[1]}_{11}(x)y + p'^{[1]}_{12}(x)z + p'^{[2]}_{13}(x) - \alpha z^2\right) \\ &\qquad + p'^{[1]}_{22}(x)p'^{[1]}_{32}(x)\left(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x)\right) \\ &\qquad + p'^{[2]}_{23}(x)p'^{[1]}_{32}y \\ &\quad = p'^{[1]}_{21}(x)p'^{[1]}_{32}(x)(p'^{[1]}_{11}(x)y + p'^{[1]}_{12}(x)z + p'^{[2]}_{13}(x)) \qquad (65)\\ &\qquad + \alpha p'^{[1]}_{21}(x)\left(p'^{[1]}_{31}(x)(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x))\right. \\ &\qquad \left. + p'^{[2]}_{33}(x)z\right) + p'^{[2]}_{23}(x)p'^{[1]}_{32}y \\ &\qquad + p'^{[1]}_{22}(x)p'^{[1]}_{32}(x)\left(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x)\right). \end{aligned}$$

The polynomial $p'^{[1]}_{32}(x)y^2z$ can also be expressed by multiplying Eq. 58 by $p'^{[1]}_{32}(x)z$ and by substituting expressions for $p'^{[1]}_{32}(x)z^2$ and yz from Eqs. 62 and 59, respectively:

$$\begin{aligned} &p'^{[1]}_{32}(x)y^2z \\ &= p'^{[1]}_{32}(x)(p'^{[1]}_{11}(x)yz + p'^{[1]}_{12}(x)z^2 + p'^{[2]}_{13}(x)z - \alpha z^3) \\ &= p'^{[1]}_{11}(x)p'^{[1]}_{32}(x)(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x)) \\ &\quad + p'^{[1]}_{12}(x)\left(-p'^{[1]}_{31}(x)(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x))\right. \\ &\quad \left. - p'^{[2]}_{33}(x)z\right) + p'^{[1]}_{13}(x)p'^{[1]}_{32}(x)z \\ &\quad + \alpha z\left(p'^{[1]}_{31}(x)(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x)) + p'^{[2]}_{33}(x)z\right) \\ &= p'^{[1]}_{11}(x)p'^{[1]}_{32}(x)(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x)) \qquad (66)\\ &\quad + p'^{[1]}_{12}(x)\left(-p'^{[1]}_{31}(x)(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z + p'^{[2]}_{23}(x))\right. \\ &\quad \left. - p'^{[2]}_{33}(x)z\right) + p'^{[1]}_{13}(x)p'^{[1]}_{32}(x)z + \\ &\quad + \alpha p'^{[1]}_{31}(x)p'^{[1]}_{21}(x)(p'^{[1]}_{21}(x)y + p'^{[1]}_{22}(x)z \\ &\quad + p'^{[2]}_{23}(x)) + \alpha p'^{[1]}_{31}p'^{[2]}_{23}(x)z + (\alpha p'^{[2]}_{33}(x) + \alpha p'^{[1]}_{31}p'^{[1]}_{22}(x))z^2. \end{aligned}$$

By combining right-hand sides of Eqs.65 and 66 we obtain the equation

$$\left(\alpha p'^{[2]}_{33}(x) + \alpha p'^{[1]}_{31}p'^{[1]}_{22}(x)\right) z^2 = f_1(y,z,1)^{[3,3,4]_x}, \quad (67)$$

where $f_1(y,z,1)^{[3,3,4]_x}$ stands for a polynomial with monomials y, z, and 1, with coefficients being polynomials of degree 3, 3, and 4 in x. Analogously, we can denote Eq. 62 and 64 as

$$p'^{[1]}_{32}(x)z^2 = f_2(y,z,1)^{[2,2,3]_x}, \quad (68)$$
$$\alpha p'^{[1]}_{31}(x)z^2 = f_3(y,z,1)^{[2,2,3]_x}. \quad (69)$$

Notice that the three polynomial Eqs. 67, 68, and 69 together with the initial Eq. 60 can be stacked into a matrix form as

$$\begin{bmatrix} f_{11}^{[2]}(x) & f_{12}^{[3]}(x) & f_{13}^{[3]}(x) & f_{14}^{[4]}(x) \\ f_{21}^{[1]}(x) & f_{22}^{[2]}(x) & f_{23}^{[2]}(x) & f_{24}^{[3]}(x) \\ f_{31}^{[1]}(x) & f_{32}^{[2]}(x) & f_{33}^{[2]}(x) & f_{34}^{[3]}(x) \\ 0 & p_{31}^{[1]}(x) & p_{32}^{[1]}(x) & p_{33}^{[2]}(x) \end{bmatrix} \begin{bmatrix} z^2 \\ y \\ z \\ 1 \end{bmatrix} = \mathtt{M}'(x) \begin{bmatrix} z^2 \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{0}, \quad (70)$$

where $\mathtt{M}'(x)$ is a 4×4 polynomial matrix which entries are polynomials in x. The solutions to the unknown x can be already obtained by finding the roots of the 8^{th} degree single variable polynomial $\det(\mathtt{M}'(x))$. However, since the computation of the determinant of a $4{\times}4$ polynomial matrix is indeed more complicated than that of a $3{\times}3$ matrix, we will continue to simplify the system by eliminating z^2.

By a proper combination of the coefficients of the monomial z^2 in Eqs. 68 and 69, *i.e.*, polynomials $p_{32}'^{[2]}(x)$ and $\alpha p_{31}'^{[2]}(x)$, we can eliminate x from these coefficients and we can derive an equation of the form

$$\begin{aligned} \gamma z^2 &= \gamma_1 f_2(y,z,1)^{[2,2,3]_x} + \gamma_2 f_3(y,z,1)^{[2,2,3]_x} \\ &= f_4(y,z,1)^{[2,2,3]_x}, \end{aligned} \quad (71)$$

where $\gamma, \gamma_1, \gamma_2 \in \mathbb{R}$ are constants and $f_4(y,z,1)^{[2,2,3]_x}$ is a linear combination of polynomials $f_2(y,z,1)^{[2,2,3]_x}$ and $f_3(y,z,1)^{[2,2,3]_x}$. Now we can rewrite the coefficient of z^2 from Eq. 67 as a combination of coefficients of z^2 in Eqs. 68 and 71 as

$$(\alpha p_{33}'^{[2]}(x) + \alpha p_{31}'^{[1]} p_{22}'^{[1]}(x)) = a^{[1]}(x) p_{32}'^{[1]}(x) + \beta\gamma, \quad (72)$$

where $a^{[1]}(x)$ is a linear polynomial in x and $\beta \in \mathbb{R}$ is a constant. Now, we can eliminate the monomial z^2 from Eq.67 by substituting a proper combination of Eqs. 68 and 71:

$$\begin{aligned} f_1(y,z,1)^{[3,3,4]_x} = a^{[1]}(x) f_2(y,z,1)^{[2,2,3]_x} \\ + \beta f_4(y,z,1)^{[2,2,3]_x}. \end{aligned} \quad (73)$$

Eq. 73 is the third polynomial equation linear in y and z, with polynomial coefficients $s_{21}^{[3]}(x), s_{22}^{[3]}(x)$ and $s_{23}^{[4]}(x)$ w.r.t. y, z, and 1.

Finally, Eqs. 60, 63 and 73 can be stacked into a matrix form

$$\begin{bmatrix} p_{31}^{[1]}(x) & p_{32}^{[1]}(x) & p_{33}^{[2]}(x) \\ s_{11}^{[3]}(x) & s_{12}^{[3]}(x) & s_{13}^{[3]}(x) \\ s_{21}^{[3]}(x) & s_{22}^{[3]}(x) & s_{23}^{[4]}(x) \end{bmatrix} \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathtt{M}(x) \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathbf{0}, \quad (74)$$

After finding the roots of the up to degree 8 polynomial $\det(\mathtt{M}(x))$ from Eq. 74, we obtain the solutions to the unknown x.

The particular solver for Case VI needs to perform 562 additions and 1156 multiplications.

A.1.7. Case VII

In the configuration of Case VII, the matrix $\mathtt{A}$ takes the form

$$\mathtt{A} = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 0 \end{bmatrix} \quad (75)$$

and Eqs. 3 can be rewritten as

$$p_{11}'^{[1]}(x)y + p_{12}'^{[1]}(x)z + p_{13}'^{[2]}(x) = y^2 + \alpha yz, \quad (76)$$
$$p_{21}'^{[1]}(x)y + p_{22}'^{[1]}(x)z + p_{23}'^{[2]}(x) = z^2 + \beta yz, \quad (77)$$
$$p_{31}'^{[1]}(x)y + p_{32}'^{[1]}(x)z + p_{33}'^{[2]}(x) = 0, \quad (78)$$

where $p_{11}'^{[1]}(x), \ldots, p_{33}'^{[2]}(x)$ are linear combinations of polynomials $p_{11}^{[1]}(x), \ldots, p_{33}^{[2]}(x)$ and $\alpha, \beta \in \mathbb{R}$ are constant.

Analogous to Case VI, only Eq. 78 is linear in y and z, *i.e.*, we need to eliminate the degree 2 monomials z^2 and yz from Eqs. 76 and 77 and use them to construct two new polynomial equations, both linear in y and z.

To obtain the first new equation, let us first multiply the Eq. 78 by y and z, respectively, to obtain expressions for polynomials $p_{31}'^{[1]}(x)y^2$ and $p_{32}'^{[1]}(x)z^2$:

$$p_{31}'^{[1]}(x)y^2 = -p_{32}'^{[1]}(x)yz - p_{33}'^{[2]}(x)y, \quad (79)$$
$$p_{32}'^{[1]}(x)z^2 = -p_{31}'^{[1]}(x)yz - p_{33}'^{[2]}(x)z. \quad (80)$$

Now, let us multiply Eq. 76 and 77 by polynomials $p_{31}'^{[1]}(x)$ and $p_{32}'^{[1]}(x)$, respectively, and substitute the expressions for $p_{31}'^{[1]}(x)y^2$ and $p_{32}'^{[1]}(x)z^2$ from Eqs. 79 and 80 into the resulting two equations:

$$\begin{aligned} &-p_{32}'^{[1]}(x)yz - p_{33}'^{[2]}(x)y + \alpha p_{31}'^{[1]}(x)yz \\ &\quad = p_{31}'^{[1]}(x)(p_{11}'^{[1]}(x)y + p_{12}'^{[1]}(x)z + p_{13}'^{[2]}(x)), \quad (81) \\ &-p_{31}'^{[1]}(x)yz - p_{33}'^{[2]}(x)z + \beta p_{32}'^{[1]}(x)yz \\ &\quad = p_{32}'^{[1]}(x)(p_{21}'^{[1]}(x)y + p_{22}'^{[1]}(x)z + p_{23}'^{[2]}(x)). \quad (82) \end{aligned}$$

Further, let us multiply Eqs. 81 and 82 by polynomials $(\beta p_{32}'^{[1]} - p_{31}'^{[1]}(x))$ and $(p_{32}'^{[1]}(x) - \alpha p_{31}'^{[1]})$, respectively, and add the two resulting equations:

$$\begin{aligned} 0 = &(\beta p_{32}'^{[1]}(x) - p_{31}'^{[1]}(x)) \quad (83) \\ &\cdot \left(p_{31}'^{[1]}(x)(p_{11}'^{[1]}(x)y + p_{12}'^{[1]}(x)z + p_{13}'^{[2]}(x)) + p_{33}'^{[2]}(x)y \right) \\ &+ (p_{32}'^{[1]}(x) - \alpha p_{31}'^{[1]}(x)) \\ &\cdot \left(p_{32}'^{[1]}(x)(p_{21}'^{[1]}(x)y + p_{22}'^{[1]}(x)z + p_{23}'^{[2]}(x)) + p_{33}'^{[2]}(x)z \right). \end{aligned}$$

Eq. 83 is the first new equation linear in y and z, with polynomial coefficients $s_{11}^{[3]}(x)$, $s_{12}^{[3]}(x)$, and $s_{13}^{[4]}(x)$ w.r.t. y, z and 1.

To obtain the second new equation, let us first derive three equations that, after 'hiding' x in the coefficient field, contain only monomials yz, z, y, and 1; this approach is analogous to Case VI. Eqs. 81 and 82 already have this special form and after rearranging of the terms can be rewritten as

$$(\alpha p'^{[1]}_{31}(x) - p'^{[1]}_{32}(x))yz = f_1(y,z,1)^{[2,2,3]_x}, \quad (84)$$
$$(\beta p'^{[1]}_{32}(x) - p'^{[1]}_{31}(x))yz = f_2(y,z,1)^{[2,2,3]_x}. \quad (85)$$

where $f_1(y,z,1)^{[2,2,3]_x}$ and $f_2(y,z,1)^{[2,2,3]_x}$ stand for polynomials with monomials y, z, and 1, with coefficients being polynomials of degree 2, 2, and 3 in x. By a proper combination of polynomials $\alpha p'^{[1]}_{31} - p'^{[1]}_{32}(x)$ and $\beta p'^{[1]}_{32} - p'^{[1]}_{31}(x)$ from Eqs. 84 and 85, we can eliminate the variable x from the coefficient of the monomial yz and we can derive the following equation:

$$\begin{aligned}\gamma yz &= \gamma_1 f_1(y,z,1)^{[2,2,3]_x} + \gamma_2 f_2(y,z,1)^{[2,2,3]_x} \\ &= f_3(y,z,1)^{[2,2,3]_x},\end{aligned} \quad (86)$$

where $\gamma, \gamma_1, \gamma_2 \in \mathbb{R}$ are constants and $f_3(y,z,1)^{[2,2,3]_x}$ is a linear combination of polynomials $f_1(y,z,1)^{[2,2,3]_x}$ and $f_2(y,z,1)^{[2,2,3]_x}$. The third equation in monomials yz, z, y, and 1 only can be obtained by expressing the polynomial $\gamma y^2 z$ in two different ways. First, let us multiply Eq. 86 by y and substitute the expression for y^2 from Eq. 76:

$$\begin{aligned}&\gamma yzy \\ &= \gamma_1 p'^{[1]}_{31}(x)\left(p'^{[1]}_{11}(x)y^2 + p'^{[1]}_{12}(x)yz + p'^{[2]}_{13}(x)y\right) \\ &\quad + \gamma_1 p'^{[2]}_{33}(x)y^2 \\ &\quad + \gamma_2 p'^{[1]}_{32}(x)\left(p'^{[1]}_{21}(x)y^2 + p'^{[1]}_{22}(x)yz + p'^{[2]}_{23}(x)y\right) \\ &\quad + \gamma_2 p'^{[2]}_{33}(x)yz \quad (87) \\ &= \gamma_1\left(p'^{[1]}_{31}(x)p'^{[1]}_{11}(x)p'^{[1]}_{11}(x)y + p'^{[1]}_{31}(x)p'^{[1]}_{11}(x)p'^{[1]}_{12}(x)z\right. \\ &\quad + p'^{[1]}_{31}(x)p'^{[1]}_{11}(x)p'^{[2]}_{13}(x) - p'^{[1]}_{31}(x)p'^{[1]}_{11}(x)\alpha yz \\ &\quad + p'^{[1]}_{31}(x)p'^{[1]}_{12}(x)yz + p'^{[1]}_{31}(x)p'^{[2]}_{13}(x)y + p'^{[2]}_{33}(x)p'^{[1]}_{11}(x)y \\ &\quad \left. + \; p'^{[2]}_{33}(x)p'^{[1]}_{12}(x)z + p'^{[2]}_{33}(x)p'^{[2]}_{13} - p'^{[2]}_{33}\alpha yz\right) \\ &\quad + \gamma_2\left(p'^{[1]}_{32}(x)p'^{[1]}_{21}(x)p'^{[1]}_{11}(x)y + p'^{[1]}_{32}(x)p'^{[1]}_{21}(x)p'^{[1]}_{12}(x)z\right. \\ &\quad + p'^{[1]}_{32}(x)p'^{[1]}_{21}(x)p'^{[2]}_{13}(x) - p'^{[1]}_{32}(x)p'^{[1]}_{21}(x)\alpha yz \\ &\quad \left. + \; p'^{[1]}_{32}(x)p'^{[1]}_{22}(x)yz + p'^{[1]}_{32}(x)p'^{[2]}_{23}(x)y + p'^{[2]}_{33}(x)yz\right) \\ &= f_4(yz,y,z,1)^{[2,3,3,4]_x},\end{aligned}$$

where $f_4(yz,y,z,1)^{[2,3,3,4]_x}$ is a polynomial in monomials yz, y, z, and 1, with coefficients being polynomials of degree 2, 3, 3, and 4 in x. The polynomial $\gamma y^2 z$ can also be expressed by multiplying Eq. 76 by γz:

$$\begin{aligned}&\gamma y^2 z \\ &= \gamma p'^{[1]}_{11}(x)yz + \gamma p'^{[1]}_{12}(x)z^2 + \gamma p'^{[1]}_{13}(x)z \quad (88) \\ &\quad - \alpha\gamma_1\left(p'^{[1]}_{31}(x)\left(p'^{[1]}_{11}(x)yz + p'^{[1]}_{12}(x)z^2 + p'^{[2]}_{13}(x)z\right)\right. \\ &\quad \left. + \; p'^{[2]}_{33}(x)yz\right) - \alpha\gamma_2\left(p'^{[1]}_{32}(x)\left(p'^{[1]}_{21}(x)yz + p'^{[1]}_{22}(x)z^2\right.\right. \\ &\quad \left.\left. + \; p'^{[2]}_{23}(x)z\right) + p'^{[2]}_{33}(x)z^2\right).\end{aligned}$$

Now, let us substitute the expression for z^2 from Eq. 77 into Eq. 88 to obtain an expression for $\gamma y^2 z$ that contains only monomials yz, z, y, and 1, with polynomial coefficients in x

$$\gamma y^2 z = f_5(yz,y,z,1)^{[2,3,3,4]_x}, \quad (89)$$

where $f_5(yz,y,z,1)^{[2,3,3,4]_x}$ is a polynomial with monomials yz, y, z, and 1, with coefficients being polynomials of degree 2, 3, 3, and 4 in x.

By combining the right-hand sides of Eqs. 87 and 89, we obtain an equation

$$p^{[2]}(x)yz = f_6(y,z,1)^{[3,3,4]_x}, \quad (90)$$

where $p^{[2]}(x)$ is a second degree polynomial in x and $f_6(y,z,1)^{[3,3,4]_x}$ is a polynomial with monomials y, z, and 1, with coefficients being polynomials of degree 3, 3, and 4 in x. Analogous to Case VI, the three polynomial Eqs. 84, 85, and 90 together with the initial Eq. 60 can be stacked into a matrix form as

$$\begin{bmatrix} f^{[1]}_{11}(x) & f^{[2]}_{12}(x) & f^{[2]}_{13}(x) & f^{[3]}_{14}(x) \\ f^{[1]}_{21}(x) & f^{[2]}_{22}(x) & f^{[2]}_{23}(x) & f^{[3]}_{24}(x) \\ f^{[2]}_{61}(x) & f^{[3]}_{62}(x) & f^{[3]}_{63}(x) & f^{[4]}_{64}(x) \\ 0 & p^{[1]}_{31}(x) & p^{[1]}_{32}(x) & p^{[2]}_{33}(x) \end{bmatrix}\begin{bmatrix} yz \\ y \\ z \\ 1 \end{bmatrix} = \mathtt{M}'(x)\begin{bmatrix} yz \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{0}, \quad (91)$$

where $\mathtt{M}'(x)$ is a 4×4 polynomial matrix which entries are polynomials in x. The solutions to the unknown x can be already obtained by finding the roots of the up to degree 8 univariate polynomial $\det(\mathtt{M}'(x))$. However, since the computation of the determinant of a 4×4 polynomial matrix is indeed more complicated than that of a 3×3 matrix, we will continue to simplify the system by eliminating yz.

Let us rewrite $p^{[2]}(x)$, the coefficient of the monomial yz from Eq. 90, as a combination of coefficients of yz from Eqs. 84 and 86 as

$$p^{[2]}(x) = a^{[1]}(x)\left(\alpha p'^{[1]}_{31} - p'^{[1]}_{32}(x)\right) + \beta\gamma, \quad (92)$$

where $a^{[1]}(x)$ is a linear polynomial in x and $\beta \in \mathbb{R}$ is a constant. Now, we can eliminate yz from 90 by substituting a proper combination of Eqs. 84 and 86 as

$$\begin{aligned}f_6(y,z,1)^{[3,3,4]_x} = a^{[1]}(x)f_1(y,z,1)^{[2,2,3]_x} \\ + \beta f_3(y,z,1)^{[2,2,3]_x}.\end{aligned} \quad (93)$$

Eq. 93 is the sought-for second new equation linear in y and z, with polynomial coefficients $s^{[3]}_{21}(x), s^{[3]}_{22}(x)$ and $s^{[4]}_{23}(x)$ w.r.t. y, z, and 1.

Finally, Eqs. 78, 83, and 93 can be stacked into a matrix form

$$\begin{bmatrix} p^{[1]}_{31}(x) & p^{[1]}_{32}(x) & p^{[2]}_{33}(x) \\ s^{[3]}_{11}(x) & s^{[3]}_{12}(x) & s^{[3]}_{13}(x) \\ s^{[3]}_{21}(x) & s^{[3]}_{22}(x) & s^{[4]}_{23}(x) \end{bmatrix}\begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathtt{M}(x)\begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathbf{0}, \quad (94)$$

After finding the roots of the up to degree 8 polynomial $\det(\mathtt{M}(x))$ from Eq. 94, we obtain the solutions to the unknown x.

The particular solver for Case VII needs to perform 878 additions and 1807 multiplications.

A.1.8. Case VIII

For the sake of completeness, let us restate the general non-generate case described in §3 of the main paper. This non-degenerate case is recognized by the full rank of matrix $\mathtt{A}$. Because of this regularity of matrix $\mathtt{A}$, we can multiply Eq. 3 by $\mathtt{A}^{-1}$, resulting in

$$\begin{bmatrix} y^2 \\ z^2 \\ yz \end{bmatrix} = \begin{bmatrix} p'_{11}(x) & p'_{12}(x) & p'_{13}(x) \\ p'_{21}(x) & p'_{22}(x) & p'_{23}(x) \\ p'_{31}(x) & p'_{32}(x) & p'_{33}(x) \end{bmatrix} \begin{bmatrix} y \\ z \\ 1 \end{bmatrix}, \tag{95}$$

where $p'_{11}(x), \ldots, p'_{33}(x)$ are linear combinations of polynomials $p_{11}(x), \ldots, p_{33}(x)$. Again, $p'_{13}(x), p'_{23}(x), p'_{33}(x)$ are quadratic polynomials in x and the remaining polynomials in Eq. 95 are linear in x. The reason why we manipulated Eqs. 2 into Eq. 95 is to express monomials y^2, z^2, and yz as polynomial functions in y, z, and 1.

Now, let us introduce three trivial identities

$$y^2 z = yzy, \tag{96}$$

$$yzz = z^2 y, \tag{97}$$

$$yzyz = y^2 z^2. \tag{98}$$

After substituting the expressions for y^2, z^2, and yz from Eq. 95 into Eqs. 96–98, we obtain the following three equations:

$$(p'_{11}(x)y + p'_{12}(x)z + p'_{13}(x))z = (p'_{31}(x)y + p'_{32}(x)z + p'_{33}(x))y, \tag{99}$$

$$(p'_{31}(x)y + p'_{32}(x)z + p'_{33}(x))z = (p'_{21}(x)y + p'_{22}(x)z + p'_{23}(x))y, \tag{100}$$

$$(p'_{31}(x)y + p'_{32}(x)z + p'_{33}(x)) \cdot (p'_{31}(x)y + p'_{32}(x)z + p'_{33}(x)) = (p'_{11}(x)y + p'_{12}(x)z + p'_{13}(x)) \cdot (p'_{21}(x)y + p'_{22}(x)z + p'_{23}(x)). \tag{101}$$

Since Eqs. 99, 100, and 101 again contain monomials y^2, z^2 and yz, we substitute expressions for y^2, z^2 and yz from Eq. 95 into Eqs. 99–101 once more. This double substitution transforms the identities from Eqs. 96–98 into the following matrix equation

$$\begin{bmatrix} s^{[2]}_{11}(x) & s^{[2]}_{12}(x) & s^{[3]}_{13}(x) \\ s^{[2]}_{21}(x) & s^{[2]}_{22}(x) & s^{[3]}_{23}(x) \\ s^{[3]}_{31}(x) & s^{[3]}_{32}(x) & s^{[4]}_{33}(x) \end{bmatrix} \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathtt{M}(x) \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \mathbf{0}, \tag{102}$$

where the upper index $[\cdot]$ denotes the maximum possible degree of the respective polynomial $s_{ij}(x)$.

As we know from elementary linear algebra, matrix Eq. 102 has a non-trivial solution if and only if the determinant of its matrix $\mathtt{M}(x)$ is zero. It can be easily inferred from the degrees of $s_{ij}(x)$ that the determinant of the 3×3 polynomial matrix $\mathtt{M}(x)$ is an up to degree 8 polynomial in x.

The solutions to the unknown x from the original Eq. 2 can now be obtained by finding the roots of the up to degree 8 polynomial $\det(\mathtt{M}(x))$. For example, these can be computed as the eigenvalues of its companion matrix [2], or more efficiently, using Sturm sequences [7] in some feasible interval. Solutions to the unknowns y and z can be obtained from $\mathtt{M}(x)$ after substituting the particular solutions for x into this matrix and solving the resulting system of two linear equations.

A.2. Symmetry and multiplicity of solutions

Besides degenerate configurations, the E3Q3 solver can handle configurations with symmetric solutions as well as solutions with multiplicity. After identifying the appropriate solver among the eight possible particular solvers of §A.1, we can proceed right to the final polynomial in one variable. This is because none of the steps of the particular solvers up to the derivation of the single variable polynomial $\det(\mathtt{M}(x)) = 0$ of degree at most 8 are influenced by the symmetry or multiplicity of the solutions. The only step that may be influenced by these configurations is the step of finding the roots this single variable polynomial. This step may be simplified for symmetric solutions or—on the other hand—it may lead to some numerical issues for solutions with multiplicity. However, it is worth noting that numerical issues that may appear in the connection with solutions with multiplicity are an inherent problem of numerical methods for finding roots of a single variable polynomials and as such is not limited to the solvers presented in this paper. This issue had to be addressed also by the previously published methods for solving the 3Q3 problem or its variants which are based on deriving a single variable polynomial, *e.g.*, by Nistér [4].

To show how to deal with symmetric solutions and with solutions with multiplicity in the E3Q3 solver, we will consider an important computer vision problem: the perspective three point absolute pose problem.

A.2.1. Perspective three point pose problem

The goal of the perspective three point pose (P3P) problem is to estimate the absolute pose of a perspective camera w.r.t. the known 3D points from three 2D-to-3D correspondences.

The P3P problem can be formulated as a 3Q3 system by considering a tetrahedron with vertices at the three known

3D points and the unknown camera center. For a calibrated camera, we can use the image points corresponding to the known 3D points to compute the angle between any pair of these points, with the camera center being the angle vertex. Such angles are sometimes called dihedral angles. In this formulation, to solve the P3P problem is to determine the lengths of the three line segments joining the camera center and the known 3D points (three legs) of the tetrahedron, given three sides of the tetrahedron base and the dihedral angles.

The solution to this formulation of the P3P problem can be found by solving 3Q3 system derived from the law of cosines

$$R_{xy}^2 = x^2 + y^2 - 2xy\cos(O_{xy}), \tag{103}$$
$$R_{xz}^2 = x^2 + z^2 - 2xz\cos(O_{xz}), \tag{104}$$
$$R_{yz}^2 = y^2 + z^2 - 2yz\cos(O_{yz}), \tag{105}$$

where R_{xy}, R_{xz}, R_{yz} are the known lengths of the three sides of the base of the tetrahedron, O_{xy}, O_{xz}, O_{yz} are the corresponding dihedral angles, and x, y, z are the unknown sides of the tetrahedron. This formulation of the P3P problem was presented in [1] and solved herein. The quadratic equations derived from the law of cosines have a specific structure as each equation contains only monomials of degree 2 in two variables. The final system has four pairs of symmetric solutions. The specific solver [1] was intended for this special form of 3Q3 equations only and it is based on the problem-specific polynomial manipulations and substitutions.

In the next, we show how to solve this problem using the proposed E3Q3 solver.

A.2.1.1 Symmetric solutions

Le us start by reshaping the system of Eqs. 103–105 into the matrix equation from Eq. 3, where the matrix $\mathtt{A}$ takes a special form

$$\mathtt{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -2\cos(O_{yz}) \end{bmatrix}. \tag{106}$$

This form ensures that the matrix $\mathtt{A}$ is regular for $\cos(O_{yz}) \neq 0$ and thus we can use the general E3Q3 solver from §A.1.8 to solve the P3P problem.

Because of the peculiar form of the matrix $\mathtt{A}$, the E3Q3 solver will arrive to an univariate polynomial in x that will take the following form:

$$a_8x^8 + a_6x^6 + a_4x^4 + a_2x^2 + a_0, \tag{107}$$

i.e., the odd order terms of the octic polynomial will vanish. This polynomial has four pairs of symmetric solutions. Note that the univariate polynomial resulting from the P3P problem will always have the form of Eq. 107, independent of the concrete values of, or any possible noise in, the input coefficients.

As we can see, in the case of the P3P problem, it is trivial to detect this special form of the final univariate polynomial. Here, we can use the substitution $w = x^2$ and by considering the octic polynomial to be a quartic in w, the roots can be found in a closed form. However, the E3Q3 solver can compute the roots of Eq. 107 using the Sturm sequences and recover the symmetric solutions without any problems even if the special form of the polynomial is quite ignored, possibly performing a few more operations than necessary.

A.2.1.2 Solutions with multiplicity

However, numerical issues may arise for problems with solutions with multiplicity. For example, let us consider an instance of the P3P problem where the base of the tetrahedron is an equilateral triangle with sides $R_{xy} = R_{xz} = R_{yz} = 2\sqrt{3}$ and the dihedral angles are equal to $\cos(O_{xy}) = \cos(O_{xz}) = \cos(O_{yz}) = \frac{20}{32}$. For the variable x, this problem has one pair of symmetric solutions $(4, -4)$ with multiplicity three and one pair of symmetric solutions $(1, -1)$ of multiplicity one.

Here, instead of one solution 4 with multiplicity three, the standard numerical methods, such as the Sturm sequences or the method based on the eigenvalues of the companion matrix, will find three different solutions $x_i = 4+\epsilon_i$ for some small ϵ_i. This numerical error may further accumulate and it may lead to larger errors in y and z. Unfortunately, barring the use of some *ad hoc* numerical limits imposed on the size of ϵ_i, such a situation is hard to detect.

In certain situations, it may be even impossible to compute the remaining variables y and z at all. This second problem appears when, after the substitution of the variable x from a solution with multiplicity into Eq. 6, the matrix $\mathtt{M}(x)$ vanishes or has rank 1. Again, this situation is hard to detect, since for solutions corrupted by some numerical error ϵ_i, the matrix $\mathtt{M}(x)$ does not vanish entirely, but only up to some error that depends on ϵ_i.

Even though the numerical issues of solutions with multiplicity are hard to detect, at least there are several ways to mitigate the problem of vanishing $\mathtt{M}(x)$. First of all, one can avoid the problem altogether by running the E3Q3 solver three times, each time 'hiding' a different variable in Eq. 3, thus avoiding the need for extracting y and z as linear solutions to Eq. 6. This effectively translates into running the original E3Q3 solver for three different permutations of the input coefficient vectors $\mathbf{c}_i$, see Eq. 2. The main drawback of this approach, besides the increased computational burden, is the fact that since the variables x, y, and z are computed separately, the orders of the particular solutions for these variables do not necessarily correspond. To obtain the solutions to the original problem, one needs to test all the permutations of the particular solutions just to keep those satisfying the original problem in Eq. 2.

A better approach is to avoid the solutions with multiplicity altogether. This can be done by transforming the original problem, *e.g.*, by taking a random linear combination of the original variables x, y, and z and by substituting them by new variables x', y' and z'. After this transformation, we will once again obtain a 3Q3 system, however, the probability that such a system will have solutions with multiplicity is very low.

To present an example of this approach, let us once again consider the P3P problem instance from the beginning of this section, where all individual variables x, y, and z have one pair of symmetric solutions $(4, -4)$ with multiplicity three and one pair of symmetric solutions $(1, -1)$ of multiplicity one,

$$\{[4,4,4],[-4,-4,-4],[4,4,1],[-4,-4,-1], [4,1,4],[-4,-1,-4],[1,4,4],[-1,-4,-4]\}. \quad (108)$$

Further, for solutions $x = 4$ and $x = -4$ the matrix $\mathtt{M}(x)$ vanishes. Thus, solving the univariate polynomial in Eq. 107 using standard methods may cause numerical problems, resulting in three different solutions $x_i = 4 + \epsilon_i$ with $\epsilon_i \approx 10^{-5}$. To avoid this problem, let us consider the following very simple linear combination of the original variables x, y, and z and substitute them for three new variables x', y' and z':

$$x' = x + 2y + 3z, \quad (109)$$
$$y' = 3x + y + z, \quad (110)$$
$$z' = x + 2y + 2z. \quad (111)$$

The original variables can be express using Eqs 109–111 as

$$x = \frac{2}{5}y' - \frac{1}{5}z', \quad (112)$$
$$y = -x' - \frac{1}{5}y' + \frac{8}{5}z', \quad (113)$$
$$z = x' - z'. \quad (114)$$

After substituting Eqs. 112–114 into Eqs. 103–105, we obtain a 3Q3 system that has four pairs of symmetric solutions:

$$\{[24,20,20],[-24,-20,-20], [21,11,17],[-21,-11,-17], [18,17,14],[-18,-17,-14], [15,17,14],[-15,-17,-14]\}. \quad (115)$$

After this simple linear transformation, we obtained a 3Q3 system that has four distinct pairs of symmetric solutions for x' and, as we can easily test, the matrix $\mathtt{M}(x')$ doesn't vanishes for these solutions. Thus, this simple substitution was sufficient to run the E3Q3 solver and solve this P3P instance without any numerical problems up to the numerical accuracy set in the Sturm sequences implementation. The solutions to the original variables can be now easily obtained by substituting solutions for x', y' and z' into Eqs. 112–114.

A.3. Hand-eye calibration problem

The third problem for which we propose a 3Q3 formulation is the problem of hand-eye calibration (HEC) with known translation. The HEC problem [5, 6] appeared for the first time in the connection with cameras mounted on robotic systems. Since then, it arose in many other fields ranging from medical applications to automotive industry. The HEC task is to find a rigid transformation $\mathtt{X}$ from the coordinate system connected with the robot's gripper to the coordinate system of a rigidly attached camera.

In this section, we will consider a variation of the HEC problem where the rotation of the gripper w.r.t. the robot global coordinate system is not known, however its translation can be measured. This variation was recently solved using the Gröbner basis method by Kukelova *et al.* [3], who formulated the problem using quaternions as a system of seven equations in seven unknowns with 16 solutions. The final Gröbner basis solver needs to perform G-J elimination of a 187×203 matrix and to compute eigenvalues for a 16×16 matrix. Here, we will show that this variation of the HEC problem can be formulated as a much simpler 3Q3 system, again using Cayley's parametrization of rotations.

First, let us define the set of rigid transformations

$$SE(3) = \left\{ \mathtt{T} \middle| \mathtt{T} = \begin{pmatrix} \mathtt{R}_\mathtt{T} & \mathbf{t}_\mathtt{T} \\ \mathbf{0}^\top & 1 \end{pmatrix}, \mathtt{R}_\mathtt{T} \in SO(3), \mathbf{t}_\mathtt{T} \in \mathbb{R}^3 \right\}. \quad (116)$$

Further, let us suppose that transformations $\mathtt{A}, \mathtt{B} \in SE(3)$ capture the motions, *i.e.*, the change of coordinate frames between two poses, of camera and robot, respectively, and that these transformations are known. Algebraically, the HEC problem can be formulated as a matrix equation

$$\mathtt{AX} = \mathtt{XB}, \quad (117)$$

where $\mathtt{X} \in SE(3)$ is the unknown hand-eye transformation. Eq. 117 can be decomposed to a matrix and a vector equation

$$\mathtt{R}_\mathtt{A}\mathtt{R}_\mathtt{X} = \mathtt{R}_\mathtt{X}\mathtt{R}_\mathtt{B}, \quad (118)$$
$$\mathtt{R}_\mathtt{A}\mathbf{t}_\mathtt{X} + \mathbf{t}_\mathtt{A} = \mathtt{R}_\mathtt{X}\mathbf{t}_\mathtt{B} + \mathbf{t}_\mathtt{X}. \quad (119)$$

It was shown in [5] that single Eq. 117 is not enough to solve for $\mathtt{X}$. At least two Eqs. 117 for two motions with different rotation axes are required. In the next, we will assume that we have performed two such camera and gripper motions

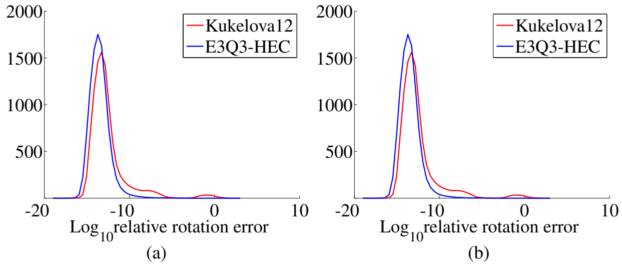

Figure 1: **HEC problem, numerical stability:** Log_{10} of (a) the relative translation and (b) rotation errors on the noise-free data for E3Q3-HEC (blue) and Kukelova12 (red).

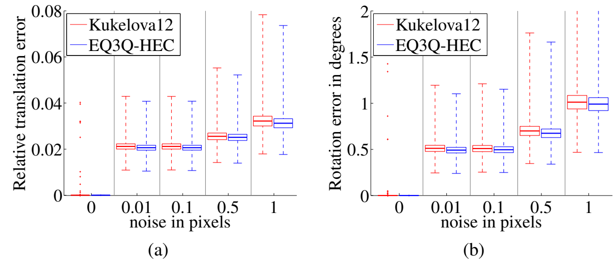

Figure 2: **HEC problem, noise experiment:** (a) Relative translation and (b) rotation errors in the presence of image noise for E3Q3-HEC (blue) and Kukelova12 (red).

$\mathtt{A}^1$,$\mathtt{A}^2$ and $\mathtt{B}^1$,$\mathtt{B}^2$, respectively, however, for various technical reasons, were unable to measure the gripper rotations $\mathtt{R}_\mathtt{B}^1$ and $\mathtt{R}_\mathtt{B}^2$.

As noted in [3], the vector part of Eq. 117 does not depend on the unknown gripper rotations $\mathtt{R}_\mathtt{B}^1$ and $\mathtt{R}_\mathtt{B}^2$ while containing all the problem's unknowns, $\mathtt{R}_\mathtt{X}$ and $\mathbf{t}_\mathtt{X}$, at the same time. In [3], the problem was solved as a system of two Eqs. 117 in seven unknowns resulting from the two motions $\mathtt{A}^i$, $\mathtt{B}^i$, $i = 1, 2$, using quaternions to represent the unknown rotation $\mathtt{R}_\mathtt{X}$. Here, we suggest to use the Cayley parametrization to obtain an even simpler 3Q3 system.

Let us parametrize the unknown rotation $\mathtt{R}_\mathtt{X}$ using three new variables $\mathbf{x} = [x, y, z]^\top$ as $\mathtt{R}_\mathtt{X}(\mathbf{x}) = \frac{1}{k}\mathtt{R}'_\mathtt{X}(\mathbf{x})$. Using this parametrization, the two Eqs. 119 can be rewritten as

$$\mathtt{R}_\mathtt{A}^i \mathbf{t}_\mathtt{X} + \mathbf{t}_\mathtt{A}^i = \tfrac{1}{k}\mathtt{R}'_\mathtt{X}(\mathbf{x})\mathbf{t}_\mathtt{B}^i + \mathbf{t}_\mathtt{X}, \quad i = 1, 2. \tag{120}$$

To transform these equations into polynomials, we need to multiply them by the denominator k. Next, we substitute vector $k\mathbf{t}_\mathtt{X}$ with a vector of three new unknowns $\hat{\mathbf{t}}_\mathtt{X}$. Now, we have six polynomial equations, three for every Eq. 119, in six unknowns x, y, z, and $\hat{\mathbf{t}}_\mathtt{X}$:

$$\mathtt{R}_\mathtt{A}^i \hat{\mathbf{t}}_\mathtt{X} + (1+x^2+y^y+z^2)\mathbf{t}_\mathtt{A}^i = \mathtt{R}'_\mathtt{X}(\mathbf{x})\mathbf{t}_\mathtt{B}^i + \hat{\mathbf{t}}_\mathtt{X}, \; i = 1, 2. \tag{121}$$

Since the system of Eqs. 121 depends on $\hat{\mathbf{t}}_\mathtt{X}$ linearly, three of these equations can be used to eliminate $\hat{\mathbf{t}}_\mathtt{X}$ from the remaining three equations, *e.g.*, using G-J elimination. After the elimination, we end up with a 3Q3 system in the unknowns x, y, and z. Once the 3Q3 problem is solved by E3Q3, we can resubstitute $k\mathbf{t}_\mathtt{X}$ back to Eqs. 120 and compute $\mathbf{t}_\mathtt{X}$ by solving this system of, now linear, equations.

Experiments. In order to gauge the numerical stability and noise sensitivity of the proposed algorithm E3Q3-HEC, we performed several experiments on synthetic data and compared the results with the state-of-the-art method Kukelova12 [3].

In the synthetic scenes used in both experiments, the camera observed a 16×16 planar calibration grid placed into the working space of a simulated robotic arm. For each scene, a random yet feasible transformation $\mathtt{X}$ was generated and the robot was instructed to perform two random motions $\mathtt{R}_\mathtt{B}^1$ and $\mathtt{R}_\mathtt{B}^2$ such that the camera rigidly connected with its gripper was still able to observe the calibration grid. Finally, we recovered the camera poses using [8] and computed the corresponding camera motions $\mathtt{A}^1$ and $\mathtt{A}^2$.

Figures 1(a–b) show the results of the numerical stability experiment on a dataset of 10K synthetic scenes with noise-free image correspondences. The relative translation error was measured as $\left\|\mathbf{t}_\mathtt{X} - \mathbf{t}_{\mathtt{X}_{gt}}\right\| / \left\|\mathbf{t}_{\mathtt{X}_{gt}}\right\|$ and rotation error as the rotation angle in the angle-axis representation of the relative rotation $\mathtt{R}_\mathtt{X}\mathtt{R}_{\mathtt{X}_{gt}}^{-1}$. Both E3Q3-HEC and Kukelova12 performed comparably well, however, Kukelova12 failed on 2% of the scenes.

In the image noise experiment, we corrupted the image correspondences with varying amount of noise prior to the camera pose recovery. The results are plotted using `boxplot` function in Figures 2(a–b). Again, Kukelova12 sometimes fails even on noise-free data. On average, E3Q3-HEC slightly outperforms Kukelova12 for all noise levels.

A.4. P4Pf—parametrization of $\mathtt{P}$

Let us recall the P4Pf problem stated in the main paper, §5.1: given four 3D scene points $\mathbf{X}_i = [x_i, y_i, z_i, 1]^\top$, $i = 1, \ldots, 4$, and four corresponding image points $\mathbf{u}_i = [u_i, v_i, 1]^\top$, $i = 1, \ldots, 4$, the task is to recover the unknown rotation $\mathtt{R}$, translation $\mathbf{t}$, and focal length $f = \frac{1}{w}$.

Each of the 2D to 3D point correspondences results in a matrix equation:

$$\begin{bmatrix} 0 & -1 & v_i \\ 1 & 0 & -u_i \\ -v_i & u_i & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ wr_{31} & wr_{32} & wr_{33} & wt_3 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix} = \mathbf{0}. \tag{122}$$

In the main paper, §5.1, we have shown how the third rows of matrix Eq. 122 for $i = 1, \ldots, 4$ can be used to parametrize the first two rows of the projection matrix $\mathtt{P}$ with the new unknowns γ_1, γ_2, and γ_3.

Here, we will use the remaining two equations from Eq. 122 to parametrize the third row of $\mathtt{P}$. For this, we have to distinguish the following two cases: if $|u_i| \le \epsilon$, for some small predetermined threshold ϵ, we will use the equation corresponding to the first row of Eq. 122; if $|v_i| \leq \epsilon$, we will use the equation corresponding to the second row. In all other cases—and these are by far the most common—we can pick one of the two equations arbitrarily, *e.g.*, let's choose the equation corresponding to the second row:

$$\begin{aligned}(p_{11}\, x_i + p_{12}\, y_i + p_{13}\, z_i + p_{14})\\ -\ u_i\,(p_{31}\, x_i + p_{32}\, y_i + p_{33}\, z_i + p_{34}) = 0.\end{aligned} \tag{123}$$

Eq. 123 contains unknowns p_{31}, p_{32}, p_{33}, and p_{34} from the third row of $\mathtt{P}$ and unknowns p_{11}, p_{12}, p_{13}, and p_{14} from the first row of $\mathtt{P}$, which we already parametrized by γ_1, γ_2, and γ_3. Again, the four 2D-3D correspondences give us four specific linear equations in the form of Eq. 123. Using the parametrization of $\mathbf{v} = [p_{11}, p_{12}, p_{13}, p_{14}, p_{21}, p_{22}, p_{23}, p_{24}]^\top$ as

$$\mathbf{v} = \gamma_1\, \mathbf{n}_1 + \gamma_2\, \mathbf{n}_2 + \gamma_3\, \mathbf{n}_3 + \mathbf{n}_4, \tag{124}$$

derived in the main paper, we can stack the four Eqs. 123 and reshape them into one matrix equation

$$\mathtt{B}\,[p_{31}, p_{32}, p_{33}, p_{34}]^\top = \mathtt{C}\,[\gamma_1, \gamma_2, \gamma_3, 1]^\top\,, \tag{125}$$

with two coefficient matrices $\mathtt{B}, \mathtt{C} \in \mathbb{R}^{4\times 4}$. If the points $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ and $\mathbf{X}_4$ are non-planar, the matrix $\mathtt{B}$ has full rank and we can rewrite Eq. 125 as

$$[p_{31}, p_{32}, p_{33}, p_{34}]^\top = \mathtt{B}^{-1}\,\mathtt{C}\,[\gamma_1, \gamma_2, \gamma_3, 1]^\top\,. \tag{126}$$

Eqs. 126 gives us a parametrization of the third row of $\mathtt{P}$ in the three unknowns γ_1, γ_2, γ_3, and a coefficient matrix $\mathtt{D} = \mathtt{B}^{-1}\,\mathtt{C}, \mathtt{D} \in \mathbb{R}^{4\times 4}$.

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