Relaxation-Based Preprocessing Techniques for Markov Random Field Inference –Supplementary Material

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1. Proof of Theorem 3

Recall several important definitions on minimum energy change for unary costs and pairwise costs given partial labeling x_S as following:

$$\begin{aligned} \delta_i(x_i) &:= \min_{y_i \neq x_i} \theta_i(y_i) - \theta_i(x_i) \\ \delta^i_{ij}(x_i, x_j) &:= \min_{y_i \neq x_i} (\theta_{ij}(y_i, x_j) - \theta_{ij}(x_i, x_j)) \\ \delta^i_{ij}(x_i, .) &:= \min_{y_i \neq x_i, z_j \in \mathcal{L}_j} (\theta_{ij}(y_i, z_j) - \theta_{ij}(x_i, z_j)) \\ \delta^{ij}_{ij}(x_i, x_j) &:= \min_{y_i \neq x_i, y_j \neq x_j} (\theta_{ij}(y_i, y_j) - \theta_{ij}(x_i, x_j)) \end{aligned} \tag{1}$$

Theorem 3. *The partial labeling x_S is persistent if:*

$$\sum_{i \in A} \delta_i(x_i) + \sum_{ij \in (A, S \setminus A)} \delta^i_{ij}(x_i, x_j) + \sum_{ij \in (A, V \setminus S)} \delta^i_{ij}(x_i, .) + \sum_{ij \in (A,A)} \delta^{ij}_{ij}(x_i, x_j) > 0, \qquad \forall A \subseteq S, A \neq \emptyset. \tag{2}$$

Proof. For arbitrary $x_{V \setminus S} \in \mathcal{L}_{V \setminus} S$ and $y_S \in \mathcal{L}_S$ such that $y_S \neq x_S$, let $A = \{i \in S \mid x_i \neq y_i\}$. Since $x_S \neq y_S$, we must have $A \neq \emptyset$. Now we have:

$$\begin{aligned} &E(y_S \oplus x_{V\setminus S}) - E(x_S \oplus x_{V \setminus S}) \\ =&\sum_{i \in A}(\theta_i(y_i) - \theta_i(x_i)) + \sum_{ij \in (A, S\setminus A)} (\theta_{ij}(y_i, x_j) - \theta_{ij}(x_i, x_j)) + \sum_{ij \in (A, V \setminus S)} (\theta_{ij}(y_i, x_j) - \theta_{ij}(x_i, x_j)) + \sum_{ij \in (A,A)} (\theta_{ij}(y_i, y_j) - \theta_{ij}(x_i, x_j)) \\ \geq& \sum_{i \in A} \delta_i(x_i) + \sum_{ij \in (A, S \setminus A)} \delta^i_{ij}(x_i, x_j) + \sum_{ij \in (A, V\setminus S)} \delta^i_{ij}(x_i, .) + \sum_{(i,j) \in (A,A)} \delta^{ij}_{ij}(x_i, x_j) \\ >&0, \end{aligned} \tag{3}$$

where the first equation above is by expanding the definition of energy function, and the second inequality is due to our definition of $\phi_i, \delta^{ij}_i, \Psi^{ij}, \Delta^{ij}_i$. □

2. Proof of Theorem 5

Theorem 5 (k-condition for S)**.** *The ILM partial labeling x_S containing at least k variables is persistent if $\forall B \subseteq S, |B| = k \geq 1$, the following inequalities hold:*

$$\sum_{i \in C} \delta_i(x_i) + \sum_{ij \in (C, B \setminus C)} \delta^i_{ij}(x_i, x_j) + \sum_{ij \in (C, V \setminus S)} \delta^i_{ij}(x_i, .) > 0, \qquad \forall C \subseteq B, C \neq \emptyset \tag{4}$$

Proof. We will show for arbitrary non-empty set $A \subseteq S$, (2) will be satisfied, hence x_S is persistent by Theorem 3. There are two cases.

Case 1: Suppose $0 < |A| \le k$, find any super set $B \supseteq A$ such that $|B| = k$, we will have:

$$\begin{aligned}
&\sum_{i\in A}\delta_i(x_i) + \sum_{ij\in(A,S\setminus A)}\delta^i_{ij}(x_i,x_j) + \sum_{ij\in(A,V\setminus S)}\delta^i_{ij}(x_i,.) + \sum_{ij\in(A,A)}\delta^{ij}_{ij}(x_i,x_j)\\
\ge&\sum_{i\in A}\delta_i(x_i) + \sum_{ij\in(A,S\setminus A)}\delta^i_{ij}(x_i,x_j) + \sum_{ij\in(A,V\setminus S)}\delta^i_{ij}(x_i,.)\\
\ge&\sum_{i\in A}\delta_i(x_i) + \sum_{ij\in(A,B\setminus A)}\delta^i_{ij}(x_i,x_j) + \sum_{ij\in(A,V\setminus S)}\delta^i_{ij}(x_i,.)\\
>&0
\end{aligned} \tag{5}$$

where the first step is due to $\delta^{ij}_{ij}(x_i,x_j) \ge 0$ for ILM labeling, the second step is due to $(A, B\setminus A) \subseteq (A, S\setminus A)$ and $\delta^i_{ij}(x_i,x_j) \ge 0$ and the last step is due to the k-condition for B and $C := A \subseteq B$.

Case 2: Suppose $|A| > k$, find all A's subset with cardinality k, i.e., $\mathcal{B} = \{B \mid B \subseteq A, |B| = k\}$ and we have:

$$\begin{aligned}
&\sum_{i\in A}\delta_i(x_i) + \sum_{ij\in(A,S\setminus A)}\delta^i_{ij}(x_i,x_j) + \sum_{ij\in(A,V\setminus S)}\delta^i_{ij}(x_i,.) + \sum_{ij\in(A,A)}\delta^{ij}_{ij}(x_i,x_j)\\
\ge&\sum_{i\in A}\delta_i(x_i) + \sum_{ij\in(A,V\setminus S)}\delta^i_{ij}(x_i,.)\\
=&\frac{1}{\binom{|A|-1}{k-1}}\cdot\binom{|A|-1}{k-1}\left(\sum_{i\in A}\delta_i(x_i) + \sum_{ij\in(A,V\setminus S)}\delta^i_{ij}(x_i,.)\right)\\
=&\frac{1}{\binom{|A|-1}{k-1}}\cdot\sum_{B\in\mathcal{B}}\left(\sum_{i\in B}\delta_i(x_i) + \sum_{ij\in(B,V\setminus S)}\delta^i_{ij}(x_i,.)\right)\\
>&0
\end{aligned} \tag{6}$$

where the first step, again, is due to $\delta^i_{ij}(x_i,x_j) \ge 0$ and $\delta^{ij}_{ij}(x_i,x_j) \ge 0$ for ILM labeling, the third equations is just a re-arrangement of terms in the double summation and the last step is due to the k-condition for B and $C := B$, note that $\sum_{ij\in(B,B\setminus B)}\delta^i_{ij}(x_i,x_j)$ vanishes since $(B, B\setminus B) = \emptyset$. □

3. Proof of Lemma 6

Lemma 6. *The ILM partial labeling x_S is persistent if we can partition S into disjoint subsets $S = \bigcup_t S_t$ and each S_t satisfies the corresponding $|S_t|$-condition.*

Proof. Again, we will show for arbitrary non-empty set $A \subseteq S$, (2) will be satisfied, hence x_S is persistent by Theorem 3.

Define A_t $A \cap S_t$, it's easy to see $A = \bigcup_t A_t$ and A_t are all disjoint. We have:

$$\begin{aligned}
&\sum_{i\in A}\delta_i(x_i) + \sum_{ij\in(A,S\setminus A)}\delta^i_{ij}(x_i,x_j) + \sum_{ij\in(A,V\setminus S)}\delta^i_{ij}(x_i,.) + \sum_{ij\in(A,A)}\delta^{ij}_{ij}(x_i,x_j)\\
\ge&\sum_{i\in A}\delta_i(x_i) + \sum_{ij\in(A,S\setminus A)}\delta^i_{ij}(x_i,x_j) + \sum_{ij\in(A,V\setminus S)}\delta^i_{ij}(x_i,.)\\
\ge&\sum_t\sum_{i\in A_t}\delta_i(x_i) + \sum_t\sum_{ij\in(A_t,S_t\setminus A_t)}\delta^i_{ij}(x_i,x_j) + \sum_t\sum_{ij\in(A_t,V\setminus S_t)}\delta^i_{ij}(x_i,.)\\
=&\sum_t\left(\sum_{i\in A_t}\delta_i(x_i) + \sum_{ij\in(A_t,S_t\setminus A_t)}\delta^i_{ij}(x_i,x_j) + \sum_{ij\in(A_t,V\setminus S_t)}\delta^i_{ij}(x_i,.)\right)\\
>&0
\end{aligned} \tag{7}$$

where the last step is due to the $|S_t|$-condition for $B := S_t$ and $C := A_t \subseteq S_t$. In the second inequality, it's easy to see $\sum_{i\in A}\delta_i(x_i) = \sum_t \sum_{i\in A_t}\delta_i(x_i)$ since $\{A_t\}$ is a disjoint partition of A, we just use the double summation to re-arrange $\delta_i(x_i)$'s. $\sum_{ij\in(A,S\backslash A)}\delta^i_{ij}(x_i,x_j) \geq \sum_t\sum_{ij\in(A_t,S_t\backslash A_t)}\delta^i_{ij}(x_i,x_j)$ and $\sum_{ij\in(A,V\backslash S)}\delta^i_{ij}(x_i,.) \geq \sum_t\sum_{ij\in(A_t,V\backslash S_t)}\delta^i_{ij}(x_i,.)$ are non-trivial and we will show next.

We want to show $\sum_{ij\in(A,S\backslash A)}\delta^i_{ij}(x_i,x_j) \geq \sum_t\sum_{ij\in(A_t,S_t\backslash A_t)}\delta^i_{ij}(x_i,x_j)$. Firstly, we have $\sum_t\sum_{ij\in(A_t,S_t\backslash A_t)}\delta^i_{ij}(x_i,x_j) = \sum_{ij\in\bigcup_t(A_t,S_t\backslash A_t)}\delta^i_{ij}(x_i,x_j)$, since A_t are all disjoint, hence edge set $(A_t,S_t\backslash A_t)$ are all disjoint as well. Next, we can see $\bigcup_t(A_t,S_t\backslash A_t) \subseteq (A,S\backslash A)$ since we have $A_t \subseteq A, S_t\backslash A_t \subseteq S\backslash A$ by their definition, hence $(A_t,S_t\backslash A_t) \subseteq (A,S\backslash A)$ for each t. Finally, we have $\delta^i_{ij}(x_i,x_j) \geq 0$ for ILM labeling, so putting things together, we have the desired inequality.

We also want to show $\sum_{ij\in(A,V\backslash S)}\delta^i_{ij}(x_i,.) \geq \sum_t\sum_{ij\in(A_t,V\backslash S_t)}\delta^i_{ij}(x_i,.)$. Again, we have $\sum_t\sum_{ij\in(A_t,V\backslash S_t)}\delta^i_{ij}(x_i,.) = \sum_{ij\in\bigcup_t(A_t,V\backslash S_t)}\delta^i_{ij}(x_i,.)$ since $(A_t,V\backslash S_t)$ are all disjoint. Next, we can show $\bigcup_t(A_t,V\backslash S_t) \supseteq (A,V\backslash S)$. Since $\forall(i,j)\in(A,V\backslash S)$, we must be able to find one A_t such that $i\in A_t$ since $A=\bigcup_t A_t$. Now it's easy to show $(i,j)\in(A_t,V\backslash S_t)$ since $V\backslash S \subseteq V\backslash S_t$. Finally, we have $\delta^i_{ij}(x_i,.) \leq 0$ by its definition (given x_i, we can pick any $y_i \neq x_i$ and let $z_j \in \arg\min_z \theta_{ij}(y_i,z)$), so putting things together, we have the desired inequality. □

4. Proof of Theorem 7

Theorem 7. $P_{DEE} \subseteq P_{\text{PR}}$ *for binary MRFs.*

Proof. Suppose not, assume $x_i=\alpha$ is the first variable which is proved to be persistent by running DEE that is not in P_{PR}. Denote the minimum energy change for unary and pairwise costs as $\delta_i(x_i), \delta^i_{ij}(x_i,x_j)$, label set as $\mathcal{L}_i$ at the time when we test if x_i is persistent. It satisfies $\delta_i(x_i)+\sum_{(i,j)\in E}\delta^i_{ij}(x_i,.)>0$. Next, considering running PR for one more iteration from its converging status to construct a constant persistent partial labeling using α. Denote the minimum energy change for this iteration as $\bar{\delta}_i(x_i), \bar{\delta}^i_{ij}(x_i,x_j), \bar{\delta}^i_{ij}(x_i,.)$ and label set as $\bar{\mathcal{L}}_i$. It's easy to see $\delta_i(x_i)=\bar{\delta}_i(x_i)$, which also depends on x_i, since both of them are checking $x_i=\alpha$. We also have $\bar{\delta}^i_{ij}(x_i,.) \geq \delta^i_{ij}(x_i,.)$ since before x_i, DEE only finds a subset of P_{PR} to be persistent, i.e., $\mathcal{L}_j \supseteq \bar{\mathcal{L}}_j, \forall j$ for the binary MRFs, which makes $\bar{\delta}^i_{ij}(x_i,.) = \min_{y_i\neq\alpha, y_j\in\bar{\mathcal{L}}_j}(\theta_{ij}(y_i,y_j)-\theta_{ij}(\alpha,y_j)) \geq \min_{y_i\neq\alpha,y_j\in\mathcal{L}_j}(\theta_{ij}(y_i,y_j)-\theta_{ij}(\alpha,y_j)) = \delta^i_{ij}(x_i,.)$. Therefore, we must have $\bar{\delta}_i(x_i)+\sum_{(i,j)\in E}\bar{\delta}^i_{ij}(x_i,.)>0$. So when we use the approximation of k-condition to test persistency, x_i will never been removed from S_α since $\{i\}$ satisfies the 1-condition, so it will never shown in the unpartitioned variable set U. When we use k-condition to test persistency. x_i will also never been removed from S_α during the shrinking procedure. Otherwise, suppose we find $B\subseteq S_\alpha$ violating k-condition and we decided to remove x_i. Recall we choose the knocked out variable with the minimum $\bar{\delta}_i(x_i)+\sum_{(i,j)\in E}\bar{\delta}^i_{ij}(x_i,.)$ value, it means we have $\bar{\delta}_i(x_i)+\sum_{(i,j)\in E}\bar{\delta}^i_{ij}(x_i,.)>0$ for all $i\in B$, which is a contradiction with B violates the k-condition. In sum, no mater what variable of PR algorithm we run, we will never remove x_i from S_α. Therefore, x_i will be proved as persistent at the end of the new iteration, which is a contraction to PR has converged. □

Remark 8. *The nice thing about* DEE *for binary MRFs is ruling out one label is equivalent to nailing down one variable. That's the key fact for us to claim $\mathcal{L}_j \supseteq \bar{\mathcal{L}}_j, \forall j$ in the proof above. For the multi-label MRFs, we cannot guarantees it. It is possible that* DEE *have ruled out some labeling from optimal labeling in $\mathcal{L}_j$ but cannot prove x_j to be persistent since we still have $|\mathcal{L}_j|>1$. However, in our experiment, we never observed that* DEE *can prove one variable to be persistent but* PR *cannot. In general,* PR *can find substantially more persistent variables than* DEE*.*

Remark 9. *Although in our hierarchical relaxation and decision problem, k-condition is always stronger than $(k+1)$-condition, we do don't necessarily have $P_{\text{PR-}k} \subseteq P_{\text{PR-}(k+1)}$, due to it depends on how to choose the knocked out variable from a violated set B. However, our experiments do indicate* PR-$(k+1)$ *can usually find significantly more persistent variables than* PR-k*.*

5. More experiments and analysis

5.1. Preprocessing multi-label MRF v.s. induced binary MRF

We showed the results on applying pre-processing to the multi-label MRF directly in Figure 1. It's conducted on color segmentation dataset (n4 version) which contains up to 12 labels. It's amazing that the mPR-based approach can achieve comparable result to KOVTUN (although still 10% less). Recalling it treats all the labels independently and purely rely on the local condition, which is much simpler than the flow-based condition KOVTUN used. mDEE is struggling in the multi-label setting, it finds very few partial optimal variables.

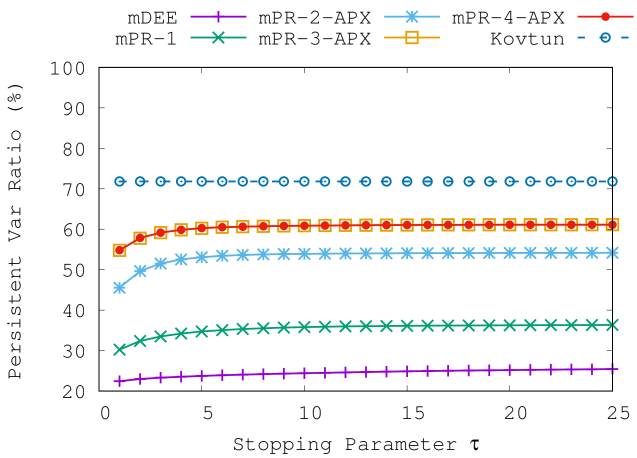

Figure 1. Preprocessing Multi-label MRFs

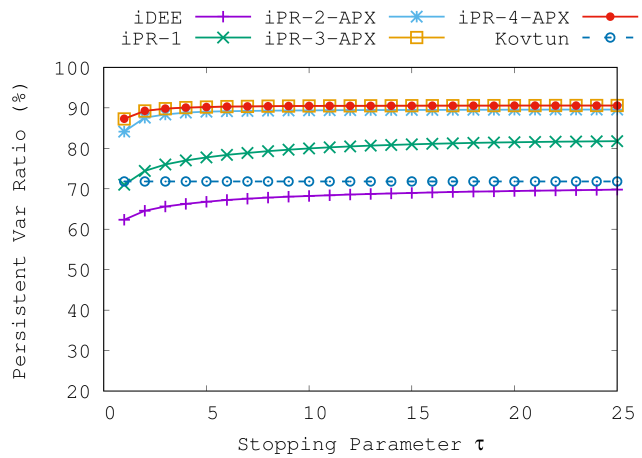

Figure 2. Preprocessing Induced Binary MRFs

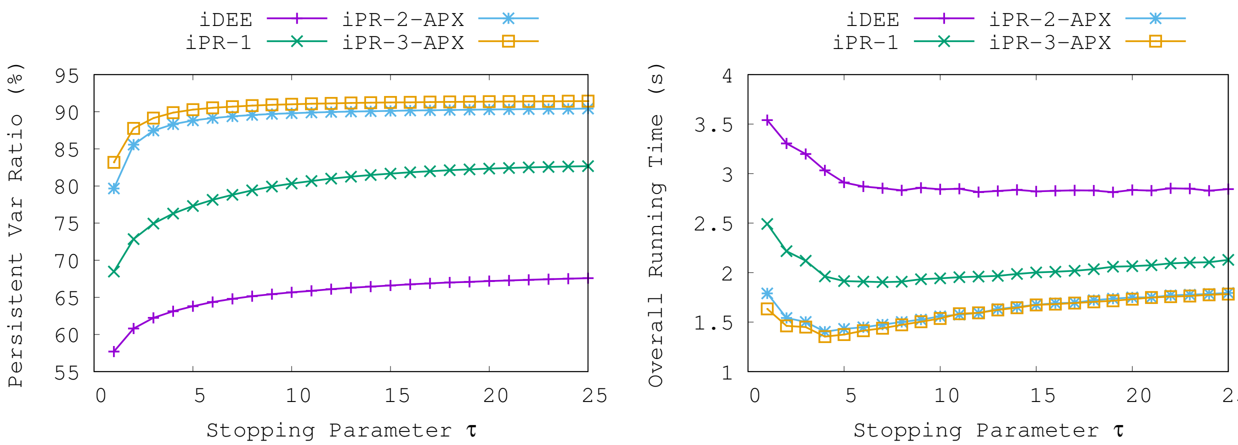

Figure 3. Persistent variables ratio vs. stopping parameter τ on Color-Seg-N4 dataset.

Comparing the left and right figure in Figure 2, we will find the advantage of applying pre-processing technique to each induced binary MRF. Note that all the iPR-based approaches all easily beat KOVTUN in this case. It indicates that although it's hard to find some global partial optimal variables, the induced binary problem are much easier. Before our approach, iDEE is the only choice suits for this per iteration pre-processing in α-expansion. Now, we provide a more general and powerful alternative.

5.2. More results for sensitivity analysis on τ

Figure 3 and 4 show the more detailed analysis of different τ values of iDEE, iPR-1, iPR-2-APX, iPR-3-APX running on the color segmentation datasets. The curves on all of the dataset look similar. Which confirms the conclusion in the main paper that the persistent var ratio converges very quickly with the growth of τ. For all these four approaches listed here, they can find the persistent variables when $\tau = 25$ no less than 1.5% compared to their converging values. In general, the overall running time decreases firstly and then increases due to it's a tradeoff between the speed and the quality of the preprocessing step. The proposed approach is not very sensitive to the choice of this stopping parameter, low total running time can be achieved in a very border range. PR-based approaches significantly outperforms DEE and other baseline methods no matter which τ we choose.

5.3. Detailed experimental results

The followed up tables provide a more detailed per-instance results with preprocessing time, flow computation time, overall running time and persistent variables ratio. (We only report average number on scene decomposition dataset due to there are 715 instances.) We can see that PR-based approaches almost win on every instance. Note that we just pick $\tau = 5$

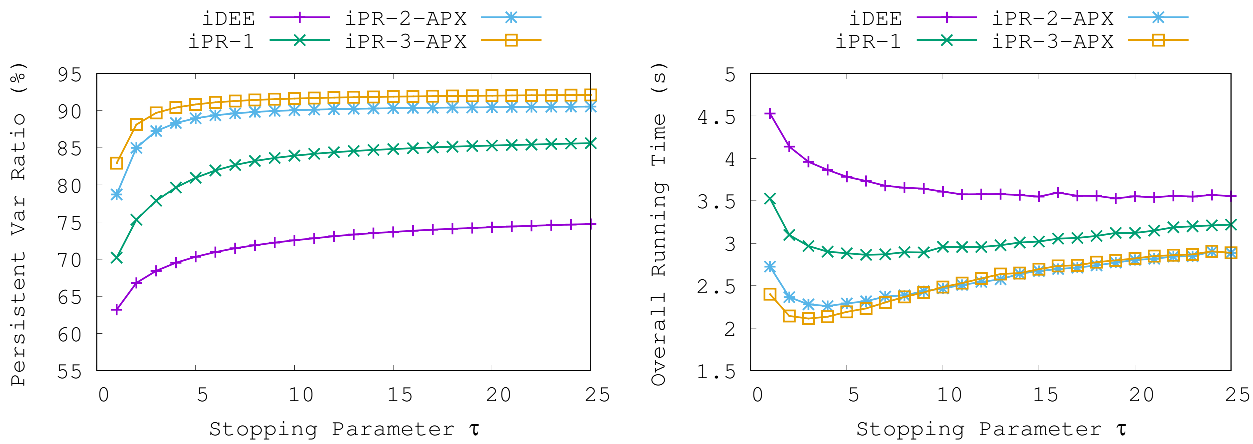

Figure 4. Persistent variables ratio vs. stopping parameter τ on Color-Seg-N8 dataset.

as an example to report the results. We conducted experiments on a variety of choices from 1 to 25 and ∞ (i.e., waiting until converge). For only very few instances, waiting until converge will spend a long time. But in general, they all achieve similar performance for $k \geq 5$. Just as we said in the sensitivity analysis section, the proposed methods are pretty robust to the choice of τ.

Table 1: Brain-MRI Dataset for $\tau = 5$

Approach	Preprocessing Time (s)	Flow Time (s)	Overall Time (s)	Speedup	Persistent Var Ratio (%)
Instance `Brain-0-9mm` (5 labels)					
α-EXP	0.0005	29.5179	30.1123	1.00	0.00
KOVTUN	5.7239	2.0030	8.2565	3.65	92.15
iDEE	0.7012	3.5980	4.8680	6.19	89.28
iPR-1	1.1698	1.0659	2.8513	10.56	97.65
iPR-2-APX	1.3974	0.6506	**2.5919**	**11.62**	99.01
iPR-3-APX	1.6068	0.6801	2.9005	10.38	99.15
iPR-4-APX	1.5891	**0.6089**	2.7399	10.99	**99.16**
Instance `Brain-1-9mm` (5 labels)					
α-EXP	0.0006	28.9777	29.5599	1.00	0.00
KOVTUN	5.5585	2.5094	8.7403	3.38	92.11
iDEE	0.5971	3.3291	4.4639	6.62	89.05
iPR-1	1.0582	0.9848	2.6253	11.26	97.64
iPR-2-APX	1.4479	0.6336	2.6252	11.26	98.99
iPR-3-APX	1.5777	0.6223	2.7479	10.76	99.12
iPR-4-APX	1.5129	**0.5853**	**2.6180**	**11.29**	**99.13**
Instance `Brain-2-9mm` (5 labels)					
α-EXP	0.0006	34.5505	35.2329	1.00	0.00
KOVTUN	8.8595	3.3385	13.0484	2.70	91.86
iDEE	0.7438	4.2960	5.7512	6.13	89.71
iPR-1	1.2384	1.1436	3.0590	11.52	97.77
iPR-2-APX	1.5239	0.7936	**2.9844**	**11.81**	99.05
iPR-3-APX	1.6548	0.6915	2.9911	11.78	99.18
iPR-4-APX	1.6875	**0.6852**	2.9992	11.75	**99.19**
Instance `Brain-3-9mm` (5 labels)					
α-EXP	0.0007	7.4870	7.6523	1.00	0.00
KOVTUN	5.3640	0.1280	5.6350	1.36	**100.00**
iDEE	0.0930	0.1834	0.4386	17.45	**100.00**
					Continued on next page

Table 1 – continued from previous page

Approach	Preprocessing Time (s)	Flow Time (s)	Overall Time (s)	Speedup	Persistent Var Ratio (%)
iPR-1	0.1432	0.1287	**0.4196**	**18.24**	**100.00**
iPR-2-APX	0.2118	0.1383	0.5037	15.19	**100.00**
iPR-3-APX	0.2250	**0.1276**	0.4943	15.48	**100.00**
iPR-4-APX	0.2254	0.1309	0.4998	15.31	**100.00**
Instance `Brain-0-5mm` (5 labels)					
α-EXP	0.0009	88.9107	90.7756	1.00	0.00
KOVTUN	12.6237	6.1703	20.6173	4.40	93.35
iDEE	2.3417	12.2558	16.5301	5.49	88.98
iPR-1	3.0442	3.0347	7.8581	11.55	97.79
iPR-2-APX	3.8545	2.0903	**7.5635**	**12.00**	99.04
iPR-3-APX	4.3486	**2.0382**	8.1504	11.14	99.17
iPR-4-APX	4.9834	2.1079	8.8157	10.30	**99.18**
Instance `Brain-1-5mm` (5 labels)					
α-EXP	0.0029	91.4609	93.3824	1.00	0.00
KOVTUN	13.0218	5.4269	20.1310	4.64	93.39
iDEE	1.9013	10.7959	14.3866	6.49	88.92
iPR-1	3.3861	3.2456	8.3935	11.13	97.80
iPR-2-APX	4.5024	2.6153	9.2742	10.07	99.04
iPR-3-APX	4.5294	**2.0818**	**8.2837**	**11.27**	99.17
iPR-4-APX	4.6707	2.1165	8.5005	10.99	**99.18**
Instance `Brain-2-5mm` (5 labels)					
α-EXP	0.0012	93.4645	95.3155	1.00	0.00
KOVTUN	12.7966	4.7755	19.0363	5.01	93.52
iDEE	1.9529	10.5398	14.2952	6.67	89.12
iPR-1	3.1350	3.0258	**7.7878**	**12.24**	97.73
iPR-2-APX	4.4257	2.3652	8.7054	10.95	99.03
iPR-3-APX	4.5862	2.1204	8.5141	11.20	99.17
iPR-4-APX	4.4486	**1.9875**	8.1466	11.70	**99.19**
Instance `Brain-3-5mm` (5 labels)					
α-EXP	0.0013	95.7588	97.6432	1.00	0.00
KOVTUN	12.6239	5.3238	19.5473	5.00	93.16
iDEE	2.1083	12.1846	16.4071	5.95	88.73
iPR-1	3.1185	2.9804	7.7995	12.52	97.85
iPR-2-APX	3.9139	2.0976	7.7210	12.65	99.08
iPR-3-APX	4.9082	2.2410	9.0263	10.82	99.20
iPR-4-APX	4.1823	**1.8582**	**7.6376**	**12.78**	**99.21**

Table 2: Color Segmentation Dataset (N4 Version) for $\tau = 5$

Approach	Preprocessing Time (s)	Flow Time (s)	Overall Time (s)	Speedup	Persistent Var Ratio (%)
Instance `clownfish` (12 labels)					
α-EXP	0.0001	11.0117	11.2568	1.00	0.00
KOVTUN	0.7680	2.0686	3.0606	3.68	73.77
iDEE	0.1386	1.2712	1.6311	6.90	86.38
iPR-1	0.2445	0.4459	0.9135	12.32	95.49
iPR-2-APX	0.3068	0.2574	**0.7887**	**14.27**	98.15
iPR-3-APX	0.3497	0.2458	0.8158	13.80	98.30
iPR-4-APX	0.3528	**0.2411**	0.8063	13.96	**98.31**
Instance `crops` (12 labels)					

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Table 2 – continued from previous page

Approach	Preprocessing Time (s)	Flow Time (s)	Overall Time (s)	Speedup	Persistent Var Ratio (%)
α-EXP	0.0001	14.0721	14.4047	1.00	0.00
KOVTUN	0.7777	4.7165	5.8195	2.48	64.44
iDEE	0.1684	2.3073	2.7720	5.20	82.52
iPR-1	0.3215	0.8315	1.4374	10.02	93.73
iPR-2-APX	0.3803	0.3885	**1.0617**	**13.57**	97.62
iPR-3-APX	0.4347	**0.3488**	1.0742	13.41	97.91
iPR-4-APX	0.4409	0.3494	1.0861	13.26	**97.92**
Instance `fourcolors` (4 labels)					
α-EXP	0.0001	1.7075	1.7472	1.00	0.00
KOVTUN	0.2423	**0.4117**	0.6915	2.53	69.16
iDEE	0.0155	1.5649	1.6145	1.08	0.23
iPR-1	0.1046	1.2016	1.3422	1.30	24.17
iPR-2-APX	0.1426	0.4985	0.6756	2.59	73.28
iPR-3-APX	0.1716	0.4621	0.6665	2.62	76.24
iPR-4-APX	0.1761	0.4511	**0.6605**	**2.65**	**76.42**
Instance `lake` (12 labels)					
α-EXP	0.0001	9.1911	9.3905	1.00	0.00
KOVTUN	0.7707	2.0279	3.0037	3.13	74.41
iDEE	0.1104	0.9078	1.2074	7.78	88.95
iPR-1	0.2127	0.4008	0.8039	11.68	95.37
iPR-2-APX	0.2769	0.2257	**0.6974**	**13.47**	98.02
iPR-3-APX	0.3198	**0.2112**	0.7234	12.98	98.19
iPR-4-APX	0.3226	0.2123	0.7234	12.98	**98.20**
Instance `palm` (12 labels)					
α-EXP	0.0001	11.6085	11.8688	1.00	0.00
KOVTUN	0.7842	3.4438	4.4894	2.64	68.44
iDEE	0.1527	4.8633	5.2478	2.26	54.86
iPR-1	0.3215	2.7908	3.3506	3.54	74.29
iPR-2-APX	0.4940	1.5906	2.3290	5.10	85.71
iPR-3-APX	0.5313	**1.3276**	**2.0944**	**5.67**	88.16
iPR-4-APX	0.5546	1.3395	2.1239	5.59	**88.19**
Instance `penguin` (8 labels)					
α-EXP	0.0001	4.7494	4.8516	1.00	0.00
KOVTUN	0.4280	**0.3012**	0.8263	5.87	**91.71**
iDEE	0.0501	1.0353	1.1810	4.11	75.61
iPR-1	0.1004	0.7650	0.9635	5.04	82.37
iPR-2-APX	0.1864	0.5120	0.7919	6.13	88.07
iPR-3-APX	0.1919	0.4410	0.7326	6.62	90.36
iPR-4-APX	0.1926	0.4329	**0.7213**	**6.73**	90.36
Instance `pfau` (12 labels)					
α-EXP	0.0001	12.4477	12.7087	1.00	0.00
KOVTUN	0.6869	13.1554	14.1369	0.90	5.55
iDEE	0.1860	6.8432	7.2792	1.75	43.33
iPR-1	0.4169	4.4423	5.1083	2.49	63.63
iPR-2-APX	0.6941	3.1020	4.0396	3.15	75.86
iPR-3-APX	0.8554	2.8141	**3.9155**	**3.25**	78.26
iPR-4-APX	0.8814	**2.8139**	3.9503	3.22	**78.29**
Instance `snail` (3 labels)					
α-EXP	0.0001	1.1219	1.1490	1.00	0.00
KOVTUN	0.1589	**0.0330**	0.2202	5.22	**97.55**

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Table 2 – continued from previous page

Approach	Preprocessing Time (s)	Flow Time (s)	Overall Time (s)	Speedup	Persistent Var Ratio (%)
iDEE	0.0211	0.1637	0.2179	5.27	83.11
iPR-1	0.0381	0.0983	0.1632	7.04	89.24
iPR-2-APX	0.0484	0.0431	**0.1180**	**9.74**	96.24
iPR-3-APX	0.0557	0.0400	0.1217	9.44	96.74
iPR-4-APX	0.0563	0.0394	0.1220	9.42	96.73
Instance `strawberry` (12 labels)					
α-EXP	0.0001	11.7457	12.0142	1.00	0.00
KOVTUN	0.6986	5.3305	6.2930	1.91	54.64
iDEE	0.1395	4.6473	5.0390	2.38	59.32
iPR-1	0.3003	2.5991	3.1464	3.82	77.54
iPR-2-APX	0.5163	1.6148	2.3899	5.03	86.39
iPR-3-APX	0.5810	**1.3912**	**2.2202**	**5.41**	88.32
iPR-4-APX	0.5908	1.4092	2.2461	5.35	**88.41**

Table 3: Color Segmentation Dataset (N8 Version) for $\tau = 5$

Approach	Preprocessing Time (s)	Flow Time (s)	Overall Time (s)	Speedup	Persistent Var Ratio (%)
Instance `clownfish` (12 labels)					
α-EXP	0.0001	15.7031	16.0336	1.00	0.00
KOVTUN	1.3051	2.7580	4.3214	3.71	73.28
iDEE	0.1992	1.0167	1.5301	10.48	93.15
iPR-1	0.3505	0.5900	1.2578	12.75	96.55
iPR-2-APX	0.4731	0.3858	**1.1771**	**13.62**	98.26
iPR-3-APX	0.5272	**0.3480**	1.1971	13.39	98.59
iPR-4-APX	0.5286	0.3489	1.1935	13.43	**98.62**
Instance `crops` (12 labels)					
α-EXP	0.0001	20.0284	20.4477	1.00	0.00
KOVTUN	1.3156	6.8882	8.6202	2.37	64.40
iDEE	0.2449	2.2717	2.9275	6.98	89.16
iPR-1	0.4669	1.0430	1.9139	10.68	95.31
iPR-2-APX	0.5811	0.5690	**1.5582**	**13.12**	97.80
iPR-3-APX	0.6512	0.5196	1.5801	12.94	98.16
iPR-4-APX	0.6644	**0.4984**	1.5657	13.06	**98.23**
Instance `fourcolors` (4 labels)					
α-EXP	0.0001	2.6387	2.6982	1.00	0.00
KOVTUN	0.4013	**0.7271**	1.1909	2.27	67.48
iDEE	0.0464	2.5394	2.6462	1.02	3.26
iPR-1	0.2163	1.5642	1.8404	1.47	43.08
iPR-2-APX	0.2779	0.8137	1.1507	2.34	73.61
iPR-3-APX	0.3351	0.7340	**1.1294**	**2.39**	77.14
iPR-4-APX	0.3563	0.7275	1.1436	2.36	**77.77**
Instance `lake` (12 labels)					
α-EXP	0.0001	13.4400	13.7211	1.00	0.00
KOVTUN	1.3099	3.1433	4.7301	2.90	73.86
iDEE	0.1689	0.9119	1.3642	10.06	93.05
iPR-1	0.3076	0.5178	1.0967	12.51	96.45
iPR-2-APX	0.4256	0.3429	**1.0407**	**13.18**	98.20
iPR-3-APX	0.4847	0.3144	1.0663	12.87	98.46
iPR-4-APX	0.4928	**0.2992**	1.0628	12.91	**98.54**

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Table 3 – continued from previous page

Approach	Preprocessing Time (s)	Flow Time (s)	Overall Time (s)	Speedup	Persistent Var Ratio (%)
Instance `palm` (12 labels)					
α-EXP	0.0001	17.1379	17.4925	1.00	0.00
KOVTUN	1.3670	5.4137	7.1379	2.45	67.95
iDEE	0.2456	5.7760	6.3707	2.75	66.62
iPR-1	0.5631	4.2506	5.1738	3.38	75.80
iPR-2-APX	0.7890	2.8098	3.9386	4.44	84.61
iPR-3-APX	0.8457	2.1140	3.3120	5.28	88.76
iPR-4-APX	0.8739	**2.0500**	**3.2671**	**5.35**	**88.96**
Instance `penguin` (8 labels)					
α-EXP	0.0001	7.3409	7.4918	1.00	0.00
KOVTUN	0.7543	**0.4608**	1.3577	5.52	**91.76**
iDEE	0.0839	1.4313	1.6677	4.49	80.11
iPR-1	0.1854	1.2173	1.5574	4.81	83.52
iPR-2-APX	0.3123	0.9932	1.4572	5.14	86.87
iPR-3-APX	0.3105	0.6887	1.1465	6.53	90.95
iPR-4-APX	0.3239	0.6726	**1.1451**	**6.54**	91.07
Instance `pfau` (12 labels)					
α-EXP	0.0001	20.1059	20.5130	1.00	0.00
KOVTUN	1.1149	16.2915	17.7468	1.16	5.64
iDEE	0.3387	9.5948	10.3246	1.99	53.47
iPR-1	0.7873	7.0523	8.2420	2.49	66.57
iPR-2-APX	1.2240	4.9666	**6.5963**	**3.11**	77.38
iPR-3-APX	1.6829	4.6373	6.7180	3.05	78.93
iPR-4-APX	1.7893	**4.5348**	6.7227	3.05	**79.21**
Instance `snail` (3 labels)					
α-EXP	0.0001	1.9664	2.0148	1.00	0.00
KOVTUN	0.2941	**0.0445**	0.3721	5.41	**97.45**
iDEE	0.0324	0.1875	0.2682	7.51	89.02
iPR-1	0.0661	0.1455	0.2615	7.70	91.79
iPR-2-APX	0.0892	0.0817	0.2217	9.09	96.44
iPR-3-APX	0.0931	0.0666	**0.2074**	**9.71**	97.26
iPR-4-APX	0.0950	0.0656	0.2083	9.67	97.36
Instance `strawberry` (12 labels)					
α-EXP	0.0001	17.4269	17.7993	1.00	0.00
KOVTUN	1.1757	8.0612	9.6044	1.85	53.82
iDEE	0.2346	6.3368	6.9496	2.56	65.13
iPR-1	0.4871	3.7356	4.5943	3.87	79.66
iPR-2-APX	0.7770	2.3417	3.4844	5.11	87.67
iPR-3-APX	0.9608	**2.0517**	**3.3714**	**5.28**	89.32
iPR-4-APX	0.9965	2.0572	3.4192	5.21	**89.59**

Table 4: Inpainting Dataset (N8 Version) for $\tau = 5$

Approach	Preprocessing Time (s)	Flow Time (s)	Overall Time (s)	Speedup	Persistent Var Ratio (%)
Instance `triplepoint4-plain-ring-inverse` (4 labels)					
α-EXP	0.0000	0.4400	0.4469	1.00	0.00
KOVTUN	0.1287	**0.1628**	0.2959	1.51	37.39
iDEE	0.0025	0.2701	0.2814	1.59	40.89
iPR-1	0.0059	0.2640	0.2765	1.62	43.99

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Table 4 – continued from previous page

Approach	Preprocessing Time (s)	Flow Time (s)	Overall Time (s)	Speedup	Persistent Var Ratio (%)
iPR-2-APX	0.0113	0.2231	0.2424	1.84	**56.02**
iPR-3-APX	0.0167	0.2159	**0.2407**	**1.86**	**56.02**
iPR-4-APX	0.0144	0.2204	0.2448	1.83	**56.02**
Instance `triplepoint4-plain-ring` (4 labels)					
α-EXP	0.0000	0.3949	0.4018	1.00	0.00
KOVTUN	0.1606	**0.0330**	0.1984	2.03	**87.38**
iDEE	0.0040	0.2393	0.2510	1.60	40.00
iPR-1	0.0061	0.2188	0.2335	1.72	47.46
iPR-2-APX	0.0151	0.1422	0.1648	2.44	67.47
iPR-3-APX	0.0133	0.1423	**0.1639**	**2.45**	67.47
iPR-4-APX	0.0138	0.1476	0.1682	2.39	67.47

Table 5: Middlebury Dataset for $\tau = 5$

Approach	Preprocessing Time (s)	Flow Time (s)	Overall Time (s)	Speedup	Persistent Var Ratio (%)
Instance `Ted` (60 labels)					
α-EXP	0.0001	401.6440	407.3978	1.00	0.00
KOVTUN	N/A	N/A	N/A	N/A	N/A
iDEE	2.7050	238.0053	244.7326	1.66	18.31
iPR-1	11.1137	178.5905	193.5722	2.10	40.96
iPR-2-APX	14.6403	125.7790	144.2356	2.82	60.24
iPR-3-APX	17.6393	121.4567	**142.8571**	**2.85**	61.47
iPR-4-APX	17.8660	**121.4136**	143.0377	2.85	**61.51**
Instance `Tsu` (16 labels)					
α-EXP	0.0001	43.4328	43.8352	1.00	0.00
KOVTUN	N/A	N/A	N/A	N/A	N/A
iDEE	0.3030	30.8764	31.4713	1.39	7.36
iPR-1	1.1738	28.7990	30.2676	1.45	14.31
iPR-2-APX	2.0666	25.8126	**28.1741**	**1.56**	24.74
iPR-3-APX	2.6680	25.4327	28.3969	1.54	**25.81**
iPR-4-APX	2.6253	**25.3058**	28.2289	1.55	**25.81**
Instance `Ven` (20 labels)					
α-EXP	0.0002	125.0321	126.2441	1.00	0.00
KOVTUN	N/A	N/A	N/A	N/A	N/A
iDEE	0.7714	85.6125	**87.2133**	**1.45**	2.43
iPR-1	2.7314	85.0040	88.5553	1.43	2.98
iPR-2-APX	5.3439	83.8337	90.0001	1.40	4.23
iPR-3-APX	7.3890	**83.6677**	91.8756	1.37	**4.46**
iPR-4-APX	7.6478	83.7015	92.1813	1.37	**4.46**
Instance `Family` (5 labels)					
α-EXP	0.0004	97.8842	98.6301	1.00	0.00
KOVTUN	N/A	N/A	N/A	N/A	N/A
iDEE	0.4242	95.6904	96.7831	1.02	4.27
iPR-1	1.3475	96.6937	98.7191	1.00	4.34
iPR-2-APX	2.1232	94.8641	97.6514	1.01	4.61
iPR-3-APX	2.2898	93.2351	96.1786	1.03	**4.63**
iPR-4-APX	2.2855	**89.4289**	**92.3023**	**1.07**	**4.63**
Instance `Pano` (7 labels)					
α-EXP	0.0004	106.8310	108.2290	1.00	0.00

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Table 5 – continued from previous page

Approach	Preprocessing Time (s)	Flow Time (s)	Overall Time (s)	Speedup	Persistent Var Ratio (%)
KOVTUN	N/A	N/A	N/A	N/A	N/A
iDEE	0.1902	107.1636	108.7092	1.00	0.00
iPR-1	0.9723	105.7037	108.1483	1.00	0.00
iPR-2-APX	2.6996	67.4241	71.5048	1.51	64.63
iPR-3-APX	2.2728	55.2445	58.7441	1.84	**67.10**
iPR-4-APX	2.2762	**53.0225**	**56.4861**	**1.92**	**67.10**
Instance `House` (256 labels)					
α-EXP	0.0001	921.9603	934.8721	1.00	0.00
KOVTUN	N/A	N/A	N/A	N/A	N/A
iDEE	4.9624	807.0047	823.6554	1.14	5.29
iPR-1	14.7405	672.4787	697.0348	1.34	5.71
iPR-2-APX	24.2710	638.4153	672.0552	1.39	7.23
iPR-3-APX	29.9763	638.7680	678.0096	1.38	7.57
iPR-4-APX	30.3008	**631.6456**	**671.1801**	**1.39**	**7.74**
Instance `Penguin` (256 labels)					
α-EXP	0.0000	262.8118	267.1431	1.00	0.00
KOVTUN	N/A	N/A	N/A	N/A	N/A
iDEE	1.3234	139.3348	143.6865	1.86	32.79
iPR-1	5.1941	120.3164	128.6085	2.08	42.92
iPR-2-APX	7.4991	98.4549	109.0183	2.45	53.85
iPR-3-APX	8.9226	96.6814	108.6613	2.46	54.70
iPR-4-APX	9.1422	**96.4438**	**108.6417**	**2.46**	**54.72**

Table 6: Scene Decomposition Dataset for $\tau = 5$

Approach	Preprocessing Time (s)	Flow Time (s)	Overall Time (s)	Speedup	Persistent Var Ratio (%)
Average over all 715 instances (8 labels)					
α-EXP	0.0000	0.0111	0.0113	1.00	0.00
KOVTUN	0.0061	0.0040	0.0103	1.10	61.66
iDEE	0.0002	0.0007	0.0011	10.27	95.08
iPR-1	0.0004	0.0005	0.0011	10.27	96.87
iPR-2-APX	0.0005	**0.0003**	**0.0010**	**11.30**	98.55
iPR-3-APX	0.0005	**0.0003**	**0.0010**	**11.30**	98.78
iPR-4-APX	0.0005	**0.0003**	**0.0010**	**11.30**	**98.82**