Partial Matchings and Growth Mapped Evolutions in Shape Spaces

Irène Kaltenmark
CMLA, ENS Cachan, CNRS,
Université Paris-Saclay, 94235 Cachan, France
irene.kaltenmark@cmla.ens-cachan.fr

Alain Trouvé
CMLA, ENS Cachan, CNRS,
Université Paris-Saclay, 94235 Cachan, France
trouve@cmla.ens-cachan.fr

Abstract

The definition of shape spaces as homogeneous spaces under the action of a group of diffeomorphisms equipped with a right invariant metric has been successful in providing theoretically sound and numerically efficient tools for registering and comparing shapes in the context of computational anatomy and leading to the so called diffeomorphometry. However, when considering not only shapes but shape evolutions or growth modeling, what could be the equivalent shape evolution spaces if any and what can be the natural group actions ? This paper proposes a principled framework in this direction on stratified shapes.

1. Introduction

Following D'Arcy Thompson's seminal work [7], the comparison of two biological shapes S_A and S_B is structured under the definition of one to one correspondences between homologous points. Since biological shapes are embedded in an ambient space, one ends up with the construction of a global diffeomorphism ϕ such that $\phi(S_A) = S_B$. From that, the first layer of the concept of shape spaces is a consistent collection of shapes and diffeomorphic mappings between them. The structure of the mapping is somewhat simple since it coincides with a group action of diffeomorphisms given by transport on shapes and this induces the *differential layer* of most shape spaces as recently formalized by Arguillière [1, 2]. The second layer is *a metric layer* inherited from the introduction of a metric structure on the mappings satisfying the triangle inequality and coming from a right invariant metric on the acting group of diffeomorphisms. This extra structure allows the development of various shape population analysis [12]. Shape spaces as Riemannian manifolds are also well adapted to the study of shape evolutions and longitudinal analysis by various methods ranging from parallel transport [6], riemannian splines [9], geodesic regression [5, 11, 4] including the inference from a population of a prototype scenario of evolution and its spatio-temporal variability [3].

However, in important situations, the starting hypothesis of the presence of homologous points between any two shapes may not be satisfied in particular during a morphogenetic process since new structures may appear (think about new cells in the biological context). In particular, partial mappings between shapes have to be considered with $\phi(S_A) \subset S_B$. A difficult question about shape evolution is the nonlinear combination of two different processes: a deformation process when a given organ is deforming through time and a growth process when the shape is evolving through a growing process involving new material. However, the situation may be more subtle: consider a shape evolution (S_t) by a pure scaling process. A first interpretation would be to explain the evolution by a continuous creation of new material on the boundary of S_t (think about the bark of a tree) preventing a matching between two non homologous rings (different year). A second interpretation, in the usual spirit of diffeomorphic registration, would be to build a one to one correspondence between the two shapes (perfect homology between points) and to explain the evolution by a pure deformation process. Note that even in this pure deformation scenario, one could consider that we have a creation of new material *stricto sensus* but the homology structure remains stable.

To deal with some of the core issues about the processing of shape evolutions in the context of growth, we propose in this paper to follow a somewhat axiomatic point of view that can be parallel to the development of the shape space point of view. In section 2, we first introduce a proper definition of the *objects* that are the atoms for the study of partial mappings and growth evolutions $(S_t)_{t\in T}$ with the notion of *growth mapped evolutions* incorporating the addition of a flow of mappings $(\phi_{s,t})_{s\leq t\in T}$ providing the homology correspondences between points within the evolution sequence. Then we define a web of morphisms between the objects organizing the relationship between the atoms. A core result will be to show that part of this web can be interpreted as coming from space-time group actions from which we can derive a metric on appropriate orbits of growth mapped evolutions. In section 3, we analyse further these orbits by showing the role of the centered evolutions corresponding to pure expansion scenarios for which the homology correspondences are trivial and on which a simple subgroup of space-time mappings can act. Finally, in section 4, we show that under reasonable regularity assumptions, any growth mapped evolution can be equipped with a tagging function, called the *birth function*, that provides a consistent stratification of the evolving shapes generalizing the idea of tree-ring dating to growth mapped evolutions.

2. Growth Evolutions

2.1. Embedded Shapes

Definition 1 (Embedded Shapes)**.**
- *An embedded shape is a pair (E, S) where E is a set called the* embedding space *or the* ambient space *and $S \subset E$.*
- *Inner Partial Mapping: For any two embedded shapes $A = (E^A, S^A)$ and $B = (E^B, S^B)$ the set $\mathrm{Hom}(A, B)$ of morphisms between A and B is given as the set of* invertible *mappings $\phi^{AB} : E^A \to E^B$ such that $\phi^{AB}(S^A) \subset S^B$.*

We check easily that if $\phi^{AB} \in \mathrm{Hom}(A, B)$ and $\phi^{BC} \in \mathrm{Hom}(B, C)$ then $\phi^{AC} \doteq \phi^{BC} \circ \phi^{AB} \in \mathrm{Hom}(A, C)$. $\mathrm{Hom}(A, B)$ will be denoted $\mathrm{Hom}_{ES}(A, B)$. The morphisms will be called ***inner partial mappings*** *between embedded shapes.*

Growth naturally induces inner partial mappings but the relations between homologous points when they exist should be preserved through time. This leads to the introduction of a set $\mathcal{L}$ of tags and of tagging functions.

Definition 2 (Tagged Embedded Shapes)**.**
- *A tagged shape over a set of tags $\mathcal{L}$ is defined as $A = (E, S, \tau)$ where (E, S) is an embedded shape and $\tau : S \to \mathcal{L}$*
- *For any two tagged embedded shapes $A = (E^A, S^A, \tau^A)$ and $B = (E^B, S^B, \tau^B)$, $\mathrm{Hom}(A, B)$ is given as the set of invertible mappings $\phi^{AB} : E^A \to E^B$ such that*
  1. *$\phi^{AB}(S^A) \subset S^B$*
  2. *$\tau^B \circ \phi^{AB} = \tau^A$ on S^A*
  3. *$\phi^{AB}(S^A) = (\tau^B)^{-1}(\tau^A(S^A))$*

The elements of $\mathrm{Hom}(A, B)$ are ***tag consistent inner partial mappings*** *between embedded shapes and $\mathrm{Hom}(A, B)$ is noted $\mathrm{Hom}_{TES}(A, B)$*

See Figure 1 for an illustration.

2.2. Growth mapped evolutions (GMEs)

A growth mapped evolution aims to model the growth scenario of an individual. The different ages of the object are represented by a collection of shapes $(S_t)_{t \in T}$ in a fixed ambient space E.

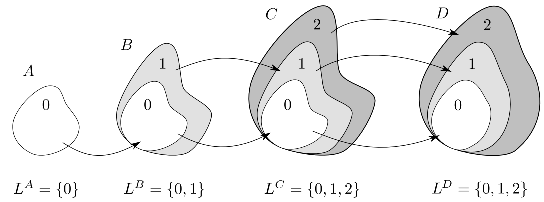

Figure 1. $L^A = \tau^A(S^A)$, $L^B = \tau^B(S^B)$, etc. denote the sets of tags involved on S^A, S^B, etc. A morphism $\phi^{A,B} \in \mathrm{Hom}_{TES}(A, B)$ must match S^A on the subset of S^B demarcated by the tag 0. The tag only defines the image set of the source shape inside the target shape. Between B and C, the tag also imposes a constraint inside the image of S^B. The points of S^B tagged by 0 are sent to the points of S^C tagged by 0 and likewise for the points tagged by 1. The arrows represent invertible mappings between the level sets of the tag functions, given by the restrictions of ϕ^{AB}, ϕ^{BC} and ϕ^{CD}. The appearance of a new tag corresponds therefore to the creation of matter uncorrelated to the previous shape. Otherwise, as between C and D, the shape is only deformed by ϕ^{CD}. We will say that the evolution is given by **pure deformation**. Even without creation ϕ^{CD} is still constrained by the tags.

Definition 3 (Growth mapped evolution of embedded shapes)**.** *A growth mapped evolution of embedded shapes in E indexed by $T \subset \mathbb{R}$ is given as $g = (T, (A_t)_{t \in T}, (\phi_{s,t})_{s \leq t \in T})$ such that*

1. *$A_t = (E, S_t)$ is an embedded shape for any $t \in T$,*
2. *$\phi_{s,t} \in \mathrm{Hom}_{ES}(A_s, A_t)$ for any $s \leq t \in T$,*
3. *$\phi_{s,t} \circ \phi_{r,s} = \phi_{r,t}$ for any $r \leq s \leq t \in T$.*

We denote $GME(T, E)$ the set of all such growth mapped evolutions.

Property (4) says that applying successively the deformations between time r and s and between time s and t gives the deformation between r and t. Note that $\phi_{t,t} = \mathrm{Id}$. We will also note $\phi_{t,s} = \phi_{s,t}^{-1}$ when $s \leq t$. $(\phi_{s,t})_{s \leq t \in T}$ will be called the *flow* of g.

Example 1 (Generation of a Circle)**.** *Let us show how two growth mapped evolutions can give two different explanations of the development of a circle. Let $T = [0, 2\pi]$, $E = \mathbb{R}^2$, and $S_t = \{(cos(\theta), sin(\theta)) | \theta \in [0, t]\}$ a collection of arcs of the unit circle, growing from the point $S_0 = \{(1, 0)\}$ to the unit circle $S_{2\pi} = \mathbb{S}^1$.*

Let us define two GMEs g^A and g^B sharing E, T and $(S_t)_{t \in T}$ as previously introduced (see Figure 2).

1. *First scenario: Complete g^A with $\phi^A_{s,t} = \mathrm{Id}$. The arcs are static. A new point appears at every time t at the extremity $(cos(t), sin(t)) \in E$. The shape is only evolving by* ***pure expansion****.*

2. *Second scenario: Complete g^B with $\phi^B_{s,t} = R_t \circ R_s^{-1}$ where R_θ is the rotation of angle θ. Here the arcs are*

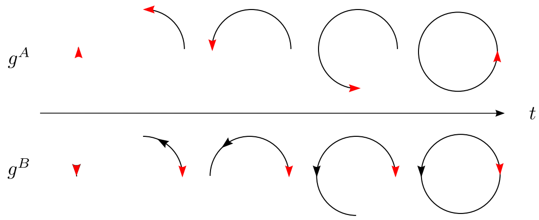

Figure 2. Illustration of Example 1. On the first row, the first scenario. Below, the second scenario. The red arrows show where the shape is expanding and therefore point to the direction in which the shape should be extending without any deformation. During the second scenario, the ambient space is rotated as implied by the black arrows.

> *gradually rotated and a new point appears at every time at the extremity $(1, 0) \in E$. The speed of the rotation canceled exactly the speed of the creation of new points so that the arcs seem static.*

In both cases, we have an expansion of the shape on its boundary, but in the second scenario, new points are all created at the same location.

2.3. Morphisms between GMEs

Morphisms between GMEs are the core of this framework. They allow us to generate a set of GMEs sharing a common growth pattern and therefore to organize them.

First, a morphism between two GMEs g^A and g^B requires a time warping ρ between T^A and T^B. Then at any time t, a spatial mapping matches the two "shapes" A and B at age t and $\rho(t)$ respectively. Moreover, these mappings must be consistent with the respective flows of each GME. This means that (assume to simplify that there is no time warping here) if we consider two points $x \in S_s^A$ and $y \in S_s^B$ at any aligned ages $s \in T$, then the spatial mappings between the two GMEs send the evolution of x in g^A, $t \mapsto x_t = \phi_{s,t}^A(x)$, to the evolution of y in g^B, $t \mapsto y_t = \phi_{s,t}^B(y)$. Finally if the GMEs are tagged, the spatial mappings must be consistent with the tags (modulo again a mapping between the tag sets of g^A and g^B).

Definition 4 (Morphism between GMEs)**.** *For any two GMEs g^A and g^B, the set $\mathrm{Hom}_{GME}(g^A, g^B)$ of morphisms between g^A and g^B is given by a time warping $\rho^{AB} : T^A \to T^B$ (non decreasing function) and a set of spatial mappings $(q_t^{AB} : S_t^A \to S_{\rho^{AB}(t)}^B)_{t \in T^A}$, such that for any $s \le t \in T^A$*

$$\begin{cases} q_t^{AB}(S_t^A) = S_{\rho^{AB}(t)}^B \\ \phi_{\rho^{AB}(s),\rho^{AB}(t)}^B \circ q_s^{AB}|_{S_t^A} = q_t^{AB} \circ \phi_{s,t}^A|_{S_t^A} . \end{cases} \tag{1}$$

2.4. Space-time Group Actions

The definition of a set of morphisms connecting GMEs that can be parallel to the first layer of the construction of shape spaces. To go further, let us show that a large subset of morphisms can be identified to a natural group action. Consider now that $T = [t_{\min}, t_{\max}]$ and E is a smooth manifold. We note $\mathrm{Diff}(T)^+$ and $\mathrm{Diff}(E)$ the groups of C^1 diffeomorphisms on T (increasing) and E respectively.

Proposition 1. *Note $G(T, E) \doteq \mathrm{Diff}(T)^+ \times \mathrm{Diff}(E)^T$ and consider for any $\Psi = (\rho, \psi = (\psi_t)_{t \in T})$, $\Psi' = (\rho', \psi' = (\psi'_t)_{t \in T}) \in G(T, E)$ the composition law defined by*

$$\Psi * \Psi' \doteq (\rho \circ \rho', (\psi_{\rho'(t)} \circ \psi'_t)_{t \in T})) \in G(T, E) . \tag{2}$$

1. *$(G(T, E), *)$ is a group with neutral element $\Psi_{\mathrm{Id}} = (\mathrm{Id}, \mathbf{Id} = (\mathrm{Id}_t)_{t \in T})$ and $\Psi^{-1} = (\rho^{-1}, (\psi^{-1}_{\rho^{-1}(t)})_{t \in T})$.*

2. *$G(T, E)$ acts on $GME(T, E)$. For any $g^A \in GME(T, E)$, any $\Psi = (\rho, \psi) \in G(T, E)$, we define $g^B \doteq (\rho, \psi) \cdot g^A$ by*

$$S_{\rho(t)}^B = \psi_t(S_t^A) \text{ and } \phi_{\rho(s),\rho(t)}^B = \psi_t \circ \phi_{s,t}^A \circ \psi_s^{-1} . \tag{3}$$

Ψ induces thus a morphism $m = (\rho, (q_t^{AB})_{t \in T}) \in \mathrm{Hom}_{GME}(g^A, g^B)$ with $q_t^{AB} = \psi_t|_{S_t^A}$.

Proof. The proof of 1) is straightforward. We just check here that the law is associative. We have indeed $((\rho, \psi) * (\rho', \psi')) * (\rho'', \psi'') = (\rho \circ \rho' \circ \rho'', (\psi_{\rho' \circ \rho''(t)} \circ \psi'_{\rho''(t)} \circ \psi''_t)_{t \in T})) = (\rho, (\psi_t)_{t \in T}) \circ (\rho' \circ \rho'', (\psi'_{\rho''(t)} \circ \psi''_t)_{t \in T}) = (\rho, \psi) * ((\rho', \psi'_t) * (\rho'', \psi''))$.

Regarding 2), if $g^C = (\rho', \psi') \cdot g^B$ then $S_{\rho' \circ \rho(t)}^C = \psi'_{\rho(t)}(S_{\rho(t)}^B) = \psi'_{\rho(t)} \circ \psi_t(S_t^A)$ and $\phi_{\rho' \circ \rho(s), \rho' \circ \rho(t)}^C = \psi'_{\rho(t)} \circ \phi_{\rho(s),\rho(t)}^B \circ (\phi'_{\rho(s)})^{-1} = (\psi'_{\rho(t)} \circ \psi_t) \circ \phi_{s,t}^A \circ (\psi_s^{-1} \circ (\psi'_{\rho(s)})^{-1}) = (\psi'_{\rho(t)} \circ \psi_t) \circ \phi_{s,t}^A \circ (\psi'_{\rho(s)} \circ \psi_s)^{-1}$. □

Example 2 (Time and Space Reparameterizations)**.** *The restrictions of $G(T, E)$ to the subgroups $\mathrm{Diff}(T)^+$ and $\mathrm{Diff}(E)$ define the basic reparameterizations in time or in space of a growth evolution. For any $\rho \in \mathrm{Diff}(T)^+$ and any $\psi \in \mathrm{Diff}(E)$, any $g \in GME(T, E)$, $g^\rho \doteq (\rho, \mathbf{Id}) \cdot g$ and $g^\psi \doteq (\mathrm{Id}, \psi) \cdot g$ are respectively given by*

$$\begin{aligned} g^\rho &= (T, (E, S_{\rho^{-1}(t)})_{t \in T}, (\phi_{\rho^{-1}(s),\rho^{-1}(t)})_{s \le t \in T}), \\ g^\psi &= (T, (E, \psi(S_t), (\psi \circ \phi_{s,t} \circ \psi^{-1})_{s \le t \in T}) . \end{aligned} \tag{4}$$

2.5. Metrics on GMEs

Let us consider $\tilde{V}$, a Reproducible Kernel Hilbert Space (RKHS) of space-time functions $\tilde{v} : T \times E \to E$, C^1 with respect to space, and H a RKHS of functions $h : T \to \mathbb{R}$ vanishing at the boundaries of T and satisfying the regularity assumptions

$$\begin{cases} \sup_{T \times E}(|\tilde{v}(t, x)| + |\partial_x \tilde{v}(t, x)|) \le K|\tilde{v}|_{\tilde{V}} \\ \sup_T(|h(t)| + |h'(t)|) \le K|h|_H . \end{cases} \tag{5}$$

We have the following theorem:

Theorem 1. *For any* $(\boldsymbol{h} = (h_s)_{s\in[0,1]}, \tilde{\boldsymbol{v}} = (\tilde{v}_s)_{s\in[0,1]}) \in L^2([0,1], H \times \tilde{V})$*, we have existence and uniqueness of the flow*

$$\begin{cases} \partial_s \psi_s(t,x) = \tilde{v}_s(\rho_s(t), \psi_s(t,x)) \\ \partial_s \rho_s(t) = h_s(\rho_s(t)) \\ \rho_0 = \mathrm{Id},\ \psi_0 = \mathbf{Id} \end{cases} \tag{6}$$

between $s = 0$ *and* $s = 1$*. If we note* $\Psi_1^{\boldsymbol{h},\tilde{\boldsymbol{v}}} = (\rho_1, \psi_1)$ *the solution at time* 1*, then*

$$G_{H\times\tilde{V}}(T,E) \doteq \{\ \Psi_1^{\boldsymbol{h},\tilde{\boldsymbol{v}}} \mid (\boldsymbol{h}, \tilde{\boldsymbol{v}}) \in L^2([0,1], H\times\tilde{V})\} \tag{7}$$

is a subgroup of $G(T,E)$ *and*

$$D(\Psi, \Psi') = \inf\{\ \|(\boldsymbol{h}, \tilde{\boldsymbol{v}})\|_2 \mid \Psi_1^{\boldsymbol{h},\tilde{\boldsymbol{v}}} * \Psi = \Psi'\} \tag{8}$$

is a right invariant distance on $G_{H\times\tilde{V}}(T,E)$*.*

Proof. (Sketch) The proof of the existence and uniqueness of the flow is an adaptation of a similar proof given in [10] where the condition (5) is an extension of the so called admissibility condition introduced in [8]. □

This right invariant distance can be seen as the Riemannian distance for the metric structure given at $(\mathrm{Id}, \mathbf{Id}) \in G_{H\times\tilde{V}}(T,E)$ by the metric on $H \times \tilde{V}$. Now, we are ready to deduce a Riemannian structure induced by the action of the space-time deformation group $G_{H\times\tilde{V}}(T,E)$ on any orbit $O_{g^0} \doteq \{\ \Phi \cdot g^0 \mid \Phi \in G_{H\times\tilde{V}}(T,E)\}$.

$$d(g, g') \doteq \inf\{D((\mathrm{Id}, \mathbf{Id}), \Psi) \mid g' = \Psi \cdot g\} \tag{9}$$

for which we deduce from standard arguments on homogeneous spaces that

Theorem 2. *The function* d *defines a pseudometric on the orbit* O_{g^0}*.*

3. Centered Growth Mapped Evolutions and Centering

Let us recall that we assume that $T = [t_{\min}, t_{\max}]$ and E is a smooth manifold. We also assume now that the flow $(\phi_{s,t})_{s\leq t\in T}$ of a GME is a set of diffeomorphisms on the ambient space (a subset of $\mathrm{Diff}(E)$).

We introduced previously the concepts of pure deformation and pure expansion to discriminate specific behaviors during a growth scenario. In the case of a **pure expansion** at all time, we will say that the GME is centered:

Definition 5. *We say that* g *is a centered growth mapped evolution of embedded shapes if* $\phi_{s,t} = \mathrm{Id}$ *for any* $s \leq t \in T$*.*

The first development of the unit circle (GME g^A) defined in Example 1 is centered. Another example is displayed in Figure 3.

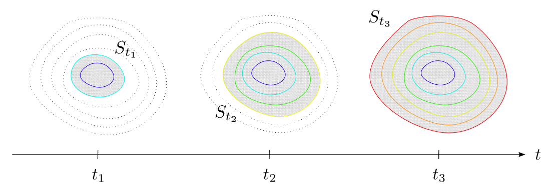

Figure 3. Evolution of a centered scenario. The colors of the curves correspond to the level sets of the tags. (The dot curves are drawn by anticipation to highlight the absence of deformation.)

Remark 1. *When a GME is centered, we get that for any* $s \leq t \in T$*,* $S_s = \phi_{s,t}(S_s) \subset S_t$ *so that the shapes form a sequence of nested sets. In particular, with* $T = [t_{\min}, t_{\max}]$ *any shape* S_t *can be seen as a subset of the end shape* $S_{t_{\max}}$*. We denote*

$$S_{all} = \cup_{s\in T} S_s = S_{t_{\max}}. \tag{10}$$

Proposition 2 (Centering a GME)**.** *If* $g = (T, (E, S_t)_{t\in T}, (\phi_{s,t})_{s\leq t\in T})$ *is a GME and* $t_c \in T$*, then* $\Phi_c = (\mathrm{Id}, (\phi_{t,t_c})_t)$ *belongs to* $G(T,E)$ *and defines a new element of* $GME(T,E)$

$$\overline{g}_c \doteq \Phi_c \cdot g. \tag{11}$$

$\overline{g}_c$ *is centered and called the centered evolution of* g *at time* t_c*.*

The action of Φ_c consists in pushing forward and pulling backward through the flow of the GME every shape S_t at time t_c. This gives for any time $t \in T$ prior to t_c the future image $\phi_{t,t_c}(S_t) \subset S_{t_c}$ of S_t at time t_c and gives a fictional inverse image $\phi_{t_c,t}^{-1}(S_t) \supset S_{t_c}$ at the younger time t_c of the more advanced shapes S_t when $t > t_c$. See an example on Figure 4.

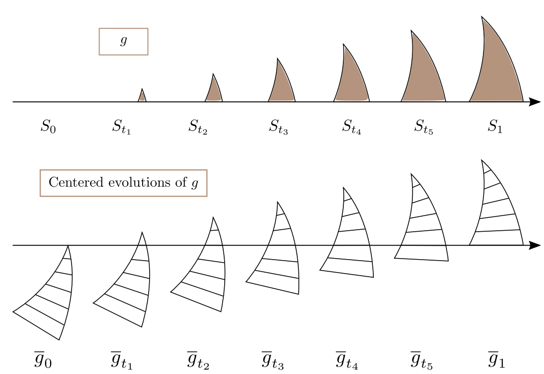

Figure 4. On the first row, a general GME g. Below, at times t_i the centered evolutions $\overline{g}_{t_i}$ of g at times t_i. We do not display as for g the trivial evolution on a time line of each $\overline{g}_{t_i}$ but only their final age with a track of every younger ages.

Remark 2. *Note that if* g *is a centered GME,* g *is its own centered evolution at any time: for any* $c \in T, \overline{g}_c = g$*. Moreover, all centered evolutions of a general GME* g *are* *equal up to an invertible spatial mapping: for any pair* $(t_c, t_{c'}) \in T$, $\phi_{t_c,t_{c'}} \in \mathrm{Diff}(E)$ *generates an element* $\Phi_{c,c'} = (\mathrm{Id}, (\phi_{t_c,t_{c'}})_t) = \Phi_{c'} * \Phi_c^{-1}$ *of* $G(T, E)$ *and* $\overline{g}_{c'} = \Phi_{c,c'} \cdot \overline{g}_c$*. The choice of* t_c *is thus meaningless.*

Remark 3 (Invertibility of the Centering)**.** *Since for any* $t_c \in T$, Φ_c *belongs to* $G(T, E)$*, the orbit of any GME* g *is generated by any centered evolution* $\overline{g}_c$ *of* g*:*

$$O_g = O_{\overline{g}_c}$$

Any general GME can thus be retrieved from one of its centered evolution and its flow. We can also generate new GMEs from centered evolutions and flows. Explicitly, given any centered evolution $\overline{g}_*$ *and any flow* $(\phi_{s,t})_{s \le t \in T}$ *of* $\mathrm{Diff}(E)$ *satisfying the transitive property (4) of Definition 3, then for any* $t_c \in T$, $\Phi_c = (\mathrm{Id}, (\phi_{t,t_c})_t)$ *belongs to* $G(T, E)$ *and* $g \doteq \Phi_c^{-1} \cdot \overline{g}_*$ *belongs to* $GME(T, E)$ *and satisfies*

$$g = (T, (E, \phi_{t_c,t}(S_t), (\phi_{s,t})_{s \le t \in T}),\ \overline{g}_c = \overline{g}_* \tag{12}$$

Proposition 3 (Stability of Centered GMEs)**.** *The image of a centered GME by an element* $\Psi = (\rho, (\psi_t)_{t \in T}) \in G(T, E)$ *is centered if and only if* $(\psi_t)_t$ *is constant in time. It defines thus an action of the subgroup* $\mathrm{Diff}(T)^+ \times \mathrm{Diff}(E) < G(T, E)$ *on the subset of centered evolutions.*

Proof. For any $g^A, g^B \in GME(E)$ such that $g^A = \Psi \cdot g^B$, g^A and g^B are centered if and only if $\phi^A_{s,t} = \phi^B_{\rho(s),\rho(t)} = \mathrm{Id}$ for any $s, t \in T^A$. Then equation (3) gives that $\psi_s = \psi_t$ for any $s, t \in T^A$ since $\phi_{s,t} = \mathrm{Id}$ (g^A is centered). □

Any $g^A, g^B \in GME(T, E)$ are in the same $G(T, E)$-orbit if and only if there exists $c \in T^A, c' \in T^B$, such that $\overline{g}^A_c$ and $\overline{g}^B_{c'}$ are in the same $\mathrm{Diff}(T)^+ \times \mathrm{Diff}(E)$-orbit. On the diagram below, $\Psi_\rho \in G(T, E)$ exists if and only if $\bar{\Psi}_\rho \in \mathrm{Diff}(T)^+ \times \mathrm{Diff}(E)$ exists.

$$\begin{array}{ccc} g^A & \xrightarrow{\Phi^A_c} & \overline{g}^A_c \\ \downarrow{\scriptstyle \Psi_\rho} & & \downarrow{\scriptstyle \bar{\Psi}_\rho} \\ g^B & \xrightarrow{\Phi^B_{c'}} & \overline{g}^B_{c'} \end{array} \tag{13}$$

We have explicitly

$$\bar{\Psi}_\rho = \Phi^B_{c'} * \Psi_\rho * (\Phi^A_c)^{-1} \tag{14}$$

$$= \left(\rho, \phi^B_{\rho(t),t_{c'}} \circ \psi_t \circ \phi^A_{t_c,t}\right) \tag{15}$$

$$= \left(\rho, \left(\psi_{\rho^{-1}(t_{c'})} \circ \phi^A_{t,\rho^{-1}(t_{c'})} \circ \psi_t^{-1}\right) \circ \psi_t \circ \phi^A_{t_c,t}\right) \tag{16}$$

$$= \left(\rho, \psi_{\rho^{-1}(t_{c'})} \circ \phi^A_{t_c,\rho^{-1}(t_{c'})}\right) . \tag{17}$$

If the centered evolutions $\overline{g}^A_c$ and $\overline{g}^B_{c'}$ are aligned with respect to ρ, meaning $t_{c'} = \rho(t_c)$, then $\bar{\Psi}_\rho = (\rho, \psi_{t_c})$. Moreover, as noticed in Remark 2, the choice of c can be changed by the action of $\mathrm{Diff}(E)$. Hence, we can always assume that $t_c = t_{\min}$ (or $t_{\max}$). Then, all centered evolutions of a same orbit are aligned for any time warping ρ.

In conclusion, we can reconstruct the $G(T, E)$-orbit of g^A from $\overline{g}^A_{t_{\min}}$, the action of $\mathrm{Diff}(T)^+ \times \mathrm{Diff}(E)$ to retrieve all the centered GMEs of the orbit and finally the set of all diffeomorphic flows $(\phi_{s,t})_{s \le t \in T}$ on the ambient space. Figure 5 illustrates this structure.

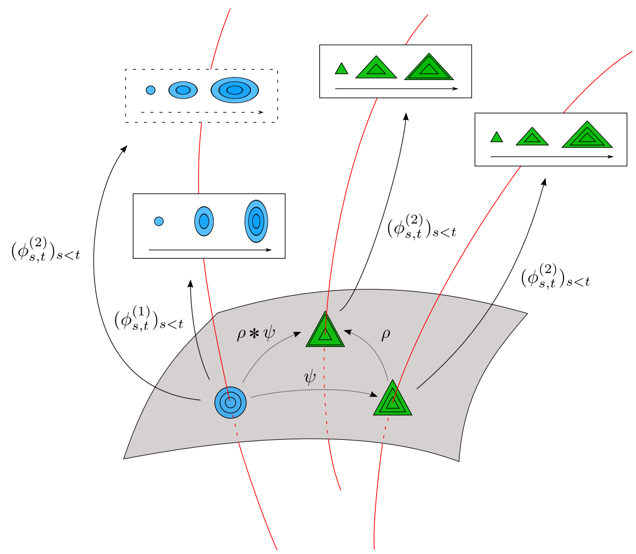

Figure 5. *Growth Evolution Space.* The gray area represents an orbit of centered GMEs under the action of $\mathrm{Diff}(T)^+ \times \mathrm{Diff}(E)$. The trivial evolution of each centered GME is implicitly displayed by a unique shape. The action of the flows of the embedding space is then represented by the vertical fibers.

4. Birth Function and Temporal Tagging

In the following, we will explain how to build canonically a temporal tagging on a large class of GMEs. Denote $\mathcal{S} = (S_t)_{t \in T}$ the time-varying shape as a single entity. We will first introduce an auxiliary function, the birth function, then formally define the birth tag and study some of their properties.

Let us start with centered GMEs. The growth process is given by pure expansion so that the shapes are simply nested and not deformed. Recall the introduction of the encompassing shape $S_{\text{all}} = \cup_{s \in T} S_s$.

Definition 6 (Birth Function of a Centered GME)**.** *When a GME is centered, one can introduce a function* $b : S_{\text{all}} \to T$*, called hereafter the* birth function *and defined as*

$$b(x) \doteq \inf\{s \in T \mid x \in S_s\}. \tag{18}$$

Note that since T *is closed,* $b(x) \in T$*. This function gives the onset of a point* x *in the evolution of* $(S_t)_{t \in T}$*.*

Likewise, the centered evolutions of a general GME generate a collection of encompassing shapes denoted

$$\overline{S}^{t}_{\text{all}} \doteq \cup_{s\in T}\phi_{s,t}(S_s) = \phi_{t_{\max},t}(S_{t_{\max}})\,. \quad (19)$$

The notion of birth function can be defined for any arbitrary GME as follows:

Definition 7 (Birth Function for a General GME)**.** *Let g be a growth mapped evolution of embedded shapes embedded in E and defined on the index set T. Let $(\phi_{s,t})_{s\le t\in T}$ be its flow. We define b the **birth function** of g as the birth function of its initial centered evolution $\overline{g}$ (see Proposition 2). Hence, b is defined on $\overline{S}^{t_{\min}}_{all} = \cup_{s\in T}\phi_{s,t_{\min}}(S_s)$ and for any $x \in \overline{S}^{t_{\min}}_{\text{all}}$ we have*

$$b(x) = \inf\{s \in T \mid \phi_{t_{\min},s}(x) \in S_s\}\,. \quad (20)$$

The birth function is thus defined on the projection $\overline{S}^{t_{\min}}_{all}$ of all shapes at time $t_{\min}$. These birth dates can now be pushed forward to the original shapes $(S_t)_{t\in T}$ to define the birth tag.

Definition 8 (Birth Tag)**.** *For any GME g, we define a canonical temporal tag called the **birth tag** and given by*

$$\tau^b_t : S_t \to T, \qquad \tau^b_t \doteq (b\circ\phi_{t,t_{\min}})_{|S_t}\,. \quad (21)$$

Note that for any $x \in S_t$, Definition 7 gives

$$\tau^b_t(x) = \inf\{s \in T \mid \phi_{t,s}(x) \in S_s\}\,. \quad (22)$$

Remark 4. *When the GME is centered, the birth function and the birth tag coincide, i.e. $\tau^b_t = b$ for all $t \in T$.*

The definition of tag consistent partial mappings (see Definition 2) enforces all points associated to one label to appear at the same time. However, the birth function does not specify if a point x that appears at time $t = b(x)$ belongs to S_t (for a centered GME). In other words, is the infinium a minimum in Definition 6 ?

Simultaneously, for a centered GME, one would like the birth function and the set of all points S_{all} to be sufficient to retrieve the evolution. This requires the following topological regularity.

Definition 9 (Right Continuity (RC))**.** *We say that a GME g is **right continuous** if for any $t \in T$ and any decreasing sequence $(t_n)_{n\ge0}$ of elements of T converging to t we have*

$$S_t = \bigcap_{n\ge0}\phi_{t_n,t}(S_{t_n})\,. \quad (23)$$

Remark 5. *When g is centered the notion of right-continuity is reduced to the property*

$$S_t = \bigcap_{n\ge0} S_{t_n}\,. \quad (24)$$

Proposition 4. *If g is a right continuous centered GME indexed by T then for any $t \in T$*

$$x \in S_t \text{ iff } b(x) \le t\,. \quad (25)$$

Proof. Indeed, if $x \in S_t$, then by definition of b, we have $b(x) \le t$. Moreover, if $b(x) < t$, then there exists $s \in T$, such that $b(x) \le s < t$ so that $x \in S_s \subset S_t$. Now if $b(x) = t$ and $x \notin S_t$, then there exists a decreasing sequence t_n of elements of T converging to t such that $x \in S_{t_n}$. Using the right continuity, we get that $x \in S_t$ which is a contradiction. Hence, if $b(x) = t$, we have $x \in S_t$. □

The proposition can be extended to any GME.

Proposition 5. *If g is a right continuous GME indexed by T then for any $t \in T$, any $x \in S_t$,*

$$x \in \phi_{s,t}(S_s) \text{ iff } \tau^b_t(x) \le s\,. \quad (26)$$

Proof. Indeed, if $x \in \phi_{s,t}(S_s)$, then by definition of τ^b_t, we have $\tau^b_t(x) \le s$. Moreover, if $\tau^b_t(x) \le s$ then $b\circ\phi_{t,t_{\min}}(x) \le s$ so that $\phi_{t,t_{\min}}(x) \in \overline{S}^{t_{\min}}_s$. This implies that $\phi_{t_{\min},s}\circ\phi_{t,t_{\min}}(x) \in S_s$ so that we get eventually $\phi_{s,t}(x) \in S_s$. □

In conclusion, given any right continuous centered GME g, the single shape S_{all} and the birth function completely describe g. Explicitly, we have for any $t \in T$,

$$S_t = \{x \in S_{\text{all}} \mid b(x) \le t\}\,. \quad (27)$$

Then, any right continuous GME can be retrieved from its initial centered evolution and its flow $(\phi_{s,t})_{s\le t\in T}$. A GME is thus characterized by these three parameters:

1. an embedded shape (E, S_{all}),
2. a birth function $b : S_{\text{all}} \to T$,
3. a flow $(\phi_{s,t})_{s\le t\in T}$.

From a modeling point of view, these two last propositions say that under the right continuous condition, if g is a centered GME, for any $x \in S_{all}$ there exists a first shape S_t containing x. Likewise if g is a general GME, if we follow any point through the flow $x_t = \phi_{t_{\min},t}(x)$ (such that $x \in S^{t_{\min}}_{\text{all}} = \phi_{t_{\max},t_{\min}}(S_{t_{\max}})$), there exists a first shape S_t containing x_t. Formally, the definition of b and τ^b can be rewritten:

$$b(x) = \min\{t \in T \mid \phi_{t_{\min},t}(x) \in S_t\}, \quad (28)$$

$$\tau^b_t(x) = \min\{s \in T \mid \phi_{t,s}(x) \in S_s\}\,. \quad (29)$$

Finally, the last proposition says that at any time t the birth tag demarcates in S_t each transported image of the preceding shapes S_s for any time $s < t \in T$

$$\phi_{s,t}(S_s) = \{x \in S_t \mid \tau^b_t(x) \le s\}\,. \quad (30)$$

All these sets contain the old points of S_t. The new points are exactly the points such that $\tau_t^b(x) = t$. This is the final ingredient to ensure that the birth function or the birth tag of a GME gives a consistent stratification coding the complete creation process during the evolution of the shape regardless of its spatial localization. Figure 6 highlights the birth tag of the two GMEs presented in Example 1.

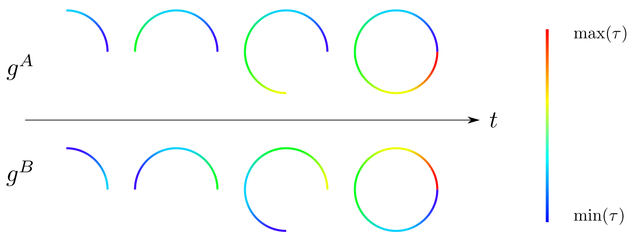

Figure 6. Consider g^A and g^B as defined in Example 1. We display the level set of their birth tag τ_t on S_t (for $t \in \{\pi/2, \pi, 3\pi/2, 2\pi\}$), on top for g^A and below for g^B. The red indicates the points that have just appeared and the blue the oldest points.

We finish this section with a technical remark.

Remark 6. *The right continuity is a necessary condition to Proposition 4 as soon as T is not a discrete set. Indeed, if there exist $t \in T$ and a decreasing sequence $t_n \to t^+$ such that $S_t \subsetneq \bigcap_{n\geq 0} S_{t_n}$, then any $x \in \bigcap_{n\geq 0} S_{t_n} \setminus S_t$ verifies $x \notin S_t$ and $b(x) \leq t$.*

4.1. Minimal Extension of a Growth Mapped Evolution of Shapes

We introduce now the notion of growth mapped evolution of tagged embedded shapes as a GME equipped with a tagging function τ such that its flow is a set of tag consistent mappings with respect to τ. The definition of morphisms between GMEs can then also be extended.

Definition 10 (TGME: Growth mapped evolution of tagged embedded shapes)**.** *A growth mapped evolution of tagged embedded shapes in E is given as $g = (T, (A_t)_{t\in T}, (\tau_t)_{t\in T}, (\phi_{s,t})_{s\leq t\in T})$ where $(T, (A_t)_{t\in T}, (\phi_{s,t})_{s\leq t\in T})$ is a GME such that for any $s \leq t \in T$, $\phi_{s,t} \in \mathrm{Hom}_{TES}(A_s, A_t)$.*

Definition 11 (Morphism between TGMEs)**.** *For any two TGMEs g^A and g^B, the set $\mathrm{Hom}_{TGME}(g^A, g^B)$ of morphisms between g^A and g^B is given by a time warping $\rho^{AB} : T^A \to T^B$, a label mapping $\eta^{AB} : L^A \to L^B$ and a set of spatial mappings $(q_t^{AB} : S_t^A \to S^B_{\rho^{AB}(t)})_{t\in T^A}$, such that for any $s, t \in T^A$*

$$(1)\ q_t^{AB}(S_t^A) = S^B_{\rho^{AB}(t)}$$

$$(2)\ \phi^B_{\rho^{AB}(s),\rho^{AB}(t)} \circ q_s^{AB}|_{S_t^A} = q_t^{AB} \circ \phi^A_{s,t}|_{S_t^A}$$

Moreover, for any $s \in T^A$ and any $x \in S_s^A$, if we denote $t = \rho^{AB}(s)$ then

$$(3)\ \tau_t^B(q_s^{AB}(x)) = \eta^{AB} \circ \tau_s^A(x')$$

for an $x' \in S_s^A$ such that $q_s^{AB}(x') = q_s^{AB}(x)$. This means that if q_s^{AB} is one to one, $\tau_t^B \circ q_s^{AB} = \eta^{AB} \circ \tau_s^A$ but this is not equivalent otherwise.

An interesting fact is that when g is a right continuous GME then the addition of a natural temporal tag, a tagging function *with values in T*, can extend it to a TGME. This temporal tag is given by the birth tag (see Definition 8). More precisely we have the proposition:

Proposition 6. *Let $g = (T, (E, (S_t)_{t\in T}), (\phi_{s,t})_{s<t\in T})$ be a right continous GME, b its birth function and $(\tau_t)_{t\in T}$ its birth tag as defined by (21), then for any times $s < t \in T$*

- *$\tau_t \leq t$,*
- *$\tau_t \circ \phi_{s,t|S_s} = \tau_s$ for $s < t \in T$,*
- *if $x \in S_t$ and $\tau_t(x) \in \tau_s(S_s)$ then $x \in \phi_{s,t}(S_s)$.*

In particular, $(\tau_t)_{t\in T}$ is a consistent tagging with respect to the flow $(\phi_{s,t})_{s<t\in T}$ and $\tilde{g} \doteq (T, (E, (S_t)_{t\in T}), (\phi_{s,t})_{s<t\in T}, (\tau_t)_{t\in T})$ is a TGME.

Proof. From (22), we get immediately that $\tau_t \leq t$ and from (21) we get that $\tau_t \circ \phi_{s,t} = b \circ \phi_{s,t_{\min}} = \tau_s$ on S_s. The last point is a direct consequence of Proposition 5. □

Definition 12 (Minimal Extension of a GME)**.** *The extension $\tilde{g}$ of a right continuous GME g defined by the previous proposition will be called the* minimal extension *of g.*

We can now state the central theorem on morphisms between minimal extensions of GMEs.

Theorem 3. *Let $m^{AB} = (\rho^{AB}, (q_s^{AB})_{s\in T^A})$ be a morphism $m^{AB} : g^A \to g^B$ between two GMEs indexed by the closed intervals T^A and T^B such that*

1. *g^A is centered,* ***right continuous****, defined on a topological embedding space E^A and S_s^A is compact for any $s \in T^A$.*
2. *E^B is a topological space and for any $t, t' \in T^B$ and $y \in S_t^B$, $t \to \phi^B_{t,t'}(y)$ is right continous.*
3. *The time warping $\rho^{AB} : T^A \to T^B$ is a increasing homeomorphism.*
4. *For any $s \in T^A$, the spatial mapping $q_s^{AB} : S_s^A \to S^B_{\rho^{AB}(s)}$ is continuous.*

Then g^B is right continuous. Moreover, there exists a morphism $\tilde{m}^{AB} = (\rho^{AB}, \eta^{AB}, (q_s^{AB})_{s\in T^A})$ between the minimal extensions $\tilde{g}^A$ and $\tilde{g}^B$ of g^A and g^B into TGMEs. We have necessarily $\eta^{AB}|_{\rho^{AB,-1}(\mathrm{Im}(\tau^B))} = \rho^{AB}$ and for any $s \in T^A$ and $x \in S_s^A$

$$\tau^B_{\rho(s)}(q_s^{AB}(x)) = \inf_{S_s^{A,y}} \rho^{AB}(\tau_s^A(z)) \tag{31}$$

where $y = q_s^{AB}(x)$ and $S_s^{A,y} \doteq \{\, z \in S_s^A \mid q_s^{AB}(z) = y\}$. If $\rho^{AB}(\mathrm{Im}(\tau^A)) = \mathrm{Im}(\tau^B)$, $\tilde{m}^{AB}$ is unique.

Proof. In the sequel, we use the notation q_s for q_s^{AB}, η for η^{AB} and ρ for ρ^{AB}. Let (t_n) be a decreasing sequence of elements in T^B converging to $t \in T^B$. Since ρ is an increasing homeomorphism, there exists a unique decreasing sequence (s_n) in T^A converging to $s \in T^A$ such that $\rho(s_n) = t_n$. However, if $y \in \bigcap_{n\geq 0} \phi^B_{t_n,t}(S^B_{t_n}) = \bigcap_{n\geq 0} \phi^B_{t_n,t} \circ q_{s_n}(S^A_{s_n})$, there exists $x_n \in S^A_{s_n}$ such that $q_{s_n}(x_n) = \phi^B_{t,t_n}(y)$. Since g^A is centered and $S^A_{s_0}$ is compact, up to the extraction of a subsequence, we can assume that x_n converges to $x \in \cap_{n\geq 0} S^A_{s_n}$. We pull forward every point at time s_0. Since the flow and the spatial map commute and again g^A is centered, we have $q_{s_0}(x_n) = q_{s_0}(\phi^A_{s_n,s_0}(x_n)) = \phi^B_{t_n,t_0}(q_{s_n}(x_n)) = \phi^B_{t_n,t_0}(y)$

On the left, we have $q_{s_0}(x_n) \to q_{s_0}(x)$ (q_{s_0} is continuous), so that $q_{s_0}(x) = \lim q_{s_n}(x_n) = \lim \phi^B_{t_n,t_0}(y) = \phi^B_{t,t_0}(y)$, the last equality coming from the assumption (2). Finally, we get $\phi^B_{t_0,t}(q_{s_0}(x)) = q_s(\phi^A_{s_0,s}(x)) = q_s(x) = y$. By right continuity of g^A, we have $x \in S_s^A$. Hence $y \in q_s(S_s^A) = S_t^B$ so that we have proved that $\bigcap_n \phi^B_{t_n,t}(S^B_{t_n}) \subset S_t^B$. Since the reverse inclusion is always true, we get that g^B is right continuous.

Since g^A is centered, note that for any $s \in T^A$, τ_s^A does not depend on s and is now denoted τ^A. Let us prove first that if $t = \rho(s)$ with $s \in T^A$, $y \in S_t^B$ and $S_s^{A,y} \doteq \{\, x \in S_s^A \mid q_s(x) = y\}$ then we have for any $t' = \rho(s')$ with $s' \in T^A$, $s' < s$ that

$$\phi^B_{t,t'}(y) \in S^B_{t'} \text{ iff } S_s^{A,y} \cap S^A_{s'} \neq \emptyset\,. \tag{32}$$

Indeed, $\phi^B_{t,t'}(y) \in S^B_{t'}$ iff there exists $x \in S^A_{s'}$ such that $y = \phi^B_{t',t}(q_{s'}(x)) = q_s(\phi^A_{s',s}(x)) = q_s(x)$ which is equivalent to $S_s^{A,y} \cap S^A_{s'} \neq \emptyset$.

Then we get if $x \in S_s^{A,y}$, $\tau_t^B(y) \leq \rho(\tau^A(x))$ and $\tau_t^B(y) \leq \inf_{x\in S_s^{A,y}} \rho(\tau^A(x))$. Now, if $s'_* = \inf\{\, s' \leq s \mid s' \in T^A,\ S_s^{A,y} \cap S^A_{s'} \neq \emptyset\,\}$ then since T^A is compact $s'_* \in T^A$ and $\tau_t^B(y) \geq \rho(s'_*)$. Moreover, by right continuity we have $S^A_{s'_*} = \cap_{u > s'_*, u\in T^A} S_u^A$ and since S_s^A is compact and $S_s^{A,y}$ closed (we assume that q_s is continuous) there exists $x_* \in S^A_{s'_*} \cap S_s^{A,y}$ so that $\tau^A(x_*) \leq s'_*$ and $q_s(x_*) = y$. Hence $\tau_t^B(y) \leq \rho(\tau^A(x_*))$ and we have proved that $\tau_t^B(y) = \inf_{x\in S_s^{A,y}} \rho(\tau^A(x))$.

Finally, let us prove that η is completely determined on $\rho^{-1}(\mathrm{Im}(\tau^B))$. With the same notations and $t'_* = \rho(s'_*)$, let us introduce $y' = \phi^B_{t,t'_*}(y) = q_{s'_*}(x_*)$ and show that $\tau^A(S^{A,y'}_{s'_*}) = s'_*$. We have $S^{A,y'}_{s'_*} \subset S_s^{A,y}$ so that $\tau^A(S^{A,y'}_{s'_*}) \subset \tau^A(S_s^{A,y})$. Now, for any $x \in S^{A,y'}_{s'_*}$, since $x \in S^A_{s'_*}$ we have $\tau^A(x) \leq s'_*$. Hence, $\rho(s'_*) = t'_* = \tau^B(y') = \eta(s'_*)$. □

The uniqueness property above allows us to transfert the birth tag of a GME on its images and retrieve the birth tags of these images:

Corollary 1. *Let g^A and g^B be two GMEs such that g^B is the image of g^A by a morphism m^{AB}. With the hypothesis of the last proposition, if τ^A is the birth tag of g^A, then the image of this tag defined on g^B by equation (31) and given by*

$$\tau^B_{\rho(s)}(q_s^{AB}(x)) = \inf_{S_s^{A,y}} \rho^{AB}(\tau_s^A(z)) \tag{33}$$

is the birth tag of g^B. Note that the definition of the image tag is here a bit more precise than in the general definition of morphisms between TGMEs (Definition 11).

Corollary 2. *For any centered GME g^A reparameterized by a time warping ρ into $g^B \doteq (\rho, \mathrm{Id}) \cdot g^A$, the birth function becomes $b^B = \rho \circ b^A$. Furthermore, a reparameterization in time between two minimal extensions $\tilde{g}^A$ and $\tilde{g}^B$ (considered as TGMEs) that preserves the birth tags must be of the type $(\rho, \rho, \mathrm{Id}) \in \mathrm{Diff}(T)^+ \times \mathrm{Diff}(L) \times \mathrm{Diff}(E)$ (where the label space is actually $L = T$).*

Remark 7. *In practice, if a centered GME is given by an encompassing embedded shape (E, S_{all}) and a birth function $b : S_{\mathrm{all}} \to T$, a reparameterization in time is equivalent to compose on the left the birth function with an invertible mapping.*

5. Conclusion

As we have seen, the notions of growth mapped evolutions and tagged growth mapped evolutions are quite effective to build a mathematical framework to handle important issues on growth modeling and analysis from a mathematical point of view. Interestingly, a Riemannian point of view can be developed on a space of growth mapped evolutions leading to the idea of *growth evolution spaces* as infinite dimensional Riemannian manifolds. The properties of such spaces can be understood thanks to the analysis of the space-time group actions acting on them and the semidirect structure of the interactions between space and time. Many interesting facts are emerging from this point of view as the key role of centered growth mapped evolutions and canonical temporal tagging opening new directions for investigation.

References

[1] S. Arguillère. The abstract setting for shape deformation analysis and lddmm methods. In *Geometric Science of Information*, pages 159–167. Springer, 2015. 1

[2] S. Arguillère. The general setting for shape analysis. *arXiv preprint arXiv:1504.01767*, 2015. 1

[3] S. Durrleman, X. Pennec, A. Trouvé, J. Braga, G. Gerig, and N. Ayache. Toward a comprehensive framework for the spatiotemporal statistical analysis of longitudinal shape data. *Int. Journal of Computer Vision*, 103(1):22–59, 2013. 1

[4] P. T. Fletcher. Geodesic regression and the theory of least squares on riemannian manifolds. *International journal of computer vision*, 105(2):171–185, 2013. 1

[5] M. Niethammer, Y. Huang, and F.-X. Vialard. Geodesic Regression for Image Time-Series. *Medical Image Computing and Computer-Assisted Intervention*, 14(Pt 2):655–662, 2011. 1

[6] A. Qiu, L. Younes, M. I. Miller, and J. G. Csernansky. Parallel transport in diffeomorphisms distinguishes the time-dependent pattern of hippocampal surface deformation due to healthy aging and the dementia of the alzheimer's type. *NeuroImage*, 40(1):68–76, 2008. 1

[7] D. Thompson. *On Growth and Form*. Dover reprint of 1942 2nd ed.(1st ed., 1917), 1992. 1

[8] A. Trouvé. Action de groupe de dimension infinie et reconnaissance de formes. *C R Acad Sci Paris Sér I Math*, 321(8):1031–1034, 1995. 4

[9] A. Trouvé and F.-X. Vialard. Shape Splines and Stochastic Shape Evolutions: A Second Order Point of View. *Quarterly of Applied Mathematics*, page 26, 2010. 1

[10] A. Trouvé and L. Younes. Local Geometry of Deformable Templates. *SIAM Journal on Mathematical Analysis*, 37(1):17, Nov. 2005. 4

[11] F.-X. Vialard, L. Risser, D. Rueckert, and C. Cotter. 3d image registration via geodesic shooting using and efficient adjoint calculation. *Journal International Journal of Computer Vision*, 97(2):229–241, April 2012. 1

[12] L. Younes. *Shapes and Diffeomorphisms*, volume 171. Series: Applied Mathematical Sciences, 2010. 1