Efficient Solvers for Minimal Problems by Syzygy-Based Reduction

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Overview

In this paper we study the problem of **automatically generating polynomial solvers** for minimal problems. The main contribution is a new method for **finding small elimination templates** by making use of the syzygies (i.e. the polynomial relations) that exist between the original equations. Using these syzygies we can essentially parameterize the set of possible elimination templates.
We evaluate our method on a wide variety of problems from geometric computer vision and show improvement compared to both handcrafted and automatically generated solvers. Furthermore we apply our method on two previously unsolved relative orientation problems.

Background

- Systems of polynomial equations occur in many geometric vision problems.

$$\begin{cases} f_1(\boldsymbol{x}) = 0 \\ \vdots \\ f_m(\boldsymbol{x}) = 0 \end{cases} \quad (1)$$

- Most common method (in Computer Vision) is the **Action Matrix** method which transforms the problem to an eigenvalue problem

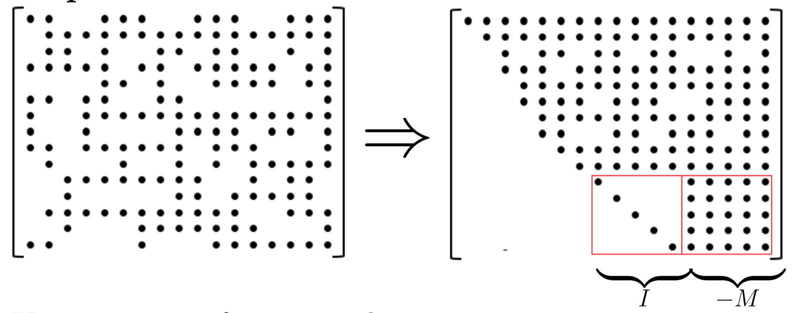
$$\alpha(\boldsymbol{x}) \begin{bmatrix} b_1(\boldsymbol{x}) \\ \vdots \\ b_K(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} & M & \end{bmatrix} \begin{bmatrix} b_1(\boldsymbol{x}) \\ \vdots \\ b_K(\boldsymbol{x}) \end{bmatrix} \quad (2)$$

$$\alpha(\boldsymbol{x}) b_i(\boldsymbol{x}) - \sum_j m_{ij} b_j(\boldsymbol{x}) = \sum_j h_{ij}(\boldsymbol{x}) f_j(\boldsymbol{x}) \quad (3)$$

Solutions to (1) are then found by eigenvalue decomposition of the matrix M.

- To find the action matrix M a so-called **elimination template** is used

$$CX = \boldsymbol{0} \quad (4)$$



where C is a constant matrix depending on the data and X is a vector of monomials.

- Each row corresponds to an equation of the type $\boldsymbol{x}^\beta f_i(\boldsymbol{x}) = 0$ where $\boldsymbol{x}^\beta$ is some monomial.
- Forming linear combinations of the rows in C allows us to find the polynomials in (3).
- Selecting which monomials to multiply the equations with to form C is difficult.
- One approach is by Kukelova et al.[33] where the elimination template is found by an iterative approach. The method alternates between expanding the set of equations and performing gaussian elimination to check if the template is sufficient.

Automatic Generator

We have implemented our approach in an automatic generator similar to that of Kukelova et al.[33].

Code is available at
`http://www.maths.lth.se/~viktorl/`

References

[33] Z. Kukelova, M. Bujnak, and T. Pajdla. Automatic generator of minimal problem solvers. In *European Conference on Computer Vision*, pages 302–315. Springer, 2008.

General problem

$$\begin{cases} f_1 = x^2 + y^2 - 1 = 0 \\ f_2 = x + ay + b = 0 \end{cases} \Rightarrow$$

Specific instance $(\mathbb{Z}_p)$

$$\begin{cases} f_1 = x^2 + y^2 - 1 = 0 \\ f_2 = x + 2y - 2 = 0 \end{cases}$$

$\Downarrow$ **GB**

$$y \begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} 8/5 & -3/5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ 1 \end{bmatrix}$$

$$y^2 - \begin{bmatrix} 8/5 & -3/5 \end{bmatrix} \begin{bmatrix} y \\ 1 \end{bmatrix} = h_1 f_1 + h_2 f_2 =$$

$$= (x + 2y - \frac{9}{5}) f_1 + \frac{1}{5}(-5x^2 - 5y^2 - x + 2y + 3) f_2$$

$\Downarrow$ **Syzygy-reduction**

$$y^2 - \begin{bmatrix} 8/5 & -3/5 \end{bmatrix} \begin{bmatrix} y \\ 1 \end{bmatrix} = \tfrac{1}{5} f_1 + \tfrac{1}{5}(-x + 2y - 2) f_2$$

$$y^2 - \begin{bmatrix} m_{11} & m_{12} \end{bmatrix} \begin{bmatrix} y \\ 1 \end{bmatrix} = c_0 f_1 + (c_1 x + c_2 y + c_3) f_2 \quad \Leftarrow$$

$\Uparrow$ **Build template**

$$\begin{array}{c|cccccc} & x^2 & xy & x & y^2 & y & 1 \\ \hline f_1 & 1 & & & 1 & & -1 \\ x f_2 & 1 & a & b & & & \\ y f_2 & & 1 & & a & b & \\ f_2 & & & 1 & & a & b \end{array}$$

$$y \begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ 1 \end{bmatrix}$$

Elimination templates

- Given h_{ij} in (3) we can form an elimination template by multiplying f_j by all monomials occurring in h_{ij}.
- However for a given problem the h_{ij} are **not unique.**
- We want to **minimize the number of monomials** (since these correspond to rows in C).
- This ambiguity is characterized by the (first) **syzygy module** of $(f_1, \ldots, f_m)$.

$$\mathrm{Syz}(f_1, \ldots, f_m) = \left\{ \boldsymbol{s} = (s_1, \ldots, s_m) \in \mathbb{K}[X]^m \;\middle|\; \sum_k s_k(\boldsymbol{x}) f_k(\boldsymbol{x}) = 0 \right\} \quad (5)$$

So for any $\boldsymbol{h}_i = (h_{i1}, \ldots, h_{im})$ which satisfies (3) we can add any element in $\mathrm{Syz}(f_1, \ldots, f_m)$ and get another elimination template.

$$\sum_j h_{ij} f_j = \sum_j (h_{ij} + s_j) f_j = \sum_j \tilde{h}_{ij} f_j \quad \forall \boldsymbol{s} \in \mathrm{Syz}(f_1, \ldots, f_m) \quad (6)$$

- Choosing the optimal $\boldsymbol{s} \in \mathrm{Syz}(f_1, \ldots, f_m)$ is a difficult problem.
- We propose a simple heuristic for simplifying the coefficients $\boldsymbol{h}_i$.
  Compute Gröbner basis G_S for the module $\mathrm{Syz}(f_1, \ldots, f_m)$ w.r.t. GRevLex-TOP.
  Compute normal form $\overline{\boldsymbol{h}_i}^{G_S}$ w.r.t. the Gröbner basis.
  Degree as a proxy for having few monomials.

Finding elimination templates

In this paper we propose the following approach for finding the elimination template for a given problem.

1. Generate an instance of the problem with coefficients in $\mathbb{Z}_p$
2. Compute a Gröbner basis $G = \{g_k\}$ and take $\{b_1, \ldots, b_k\}$ as the standard monomials.
3. Form the polynomials

$$\alpha b_i - \overline{\alpha b_i}^G = \alpha b_i - \sum_j m_{ij} b_j \quad (7)$$

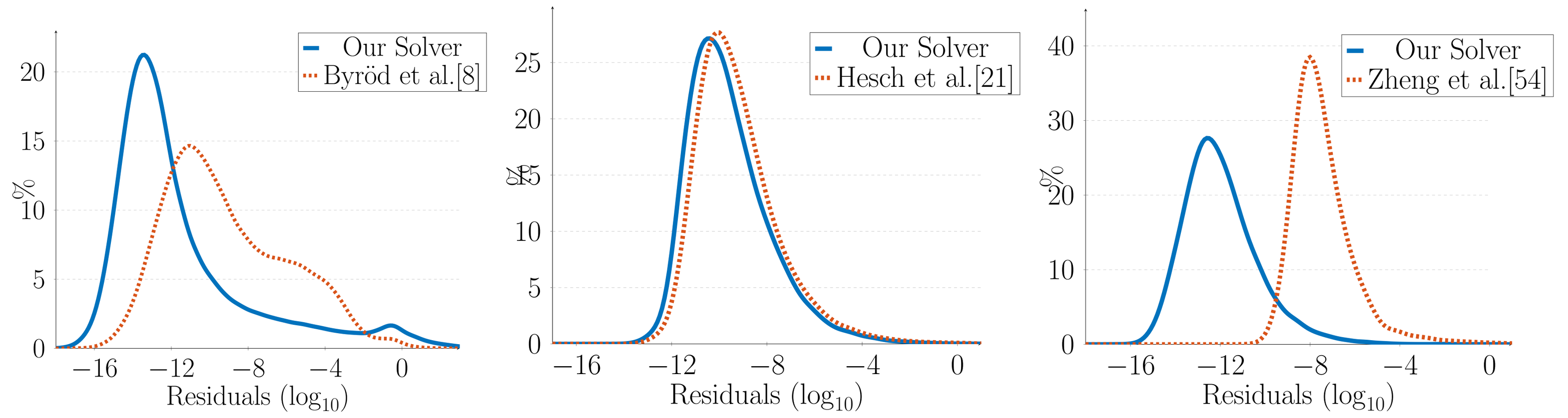
and find coefficients h_{ij} satisfying (3) by division with G.

4. Compute Gröbner basis G_S for $\mathrm{Syz}(f_1, \ldots, f_m)$ and form $\tilde{\boldsymbol{h}}_i = \overline{\boldsymbol{h}_i}^{G_S}$
5. Build template by multiplying the equations with the monomials from $\tilde{h}_{ij}$.

Each of these steps can be performed efficiently in a few lines of Macaulay2 code.

Computer Vision and Pattern Recognition, 2017

Experimental evaluation



Histograms of residual errors for 5,000 runs – from left to right – *image stitching with unknown focal length and radial distortion* [8, 39] , the *optimal PnP*-method of Hesch et al.[21] and the *optimal PnP*-method of Zheng et al.[54].

Problem	Original		Proposed generator	
	Author	template size	no reduction step	with reduction step
Rel. pose 5pt	Stewénius et al.[47]	$\mathbf{10 \times 20}$	$\mathbf{10 \times 20}$	$\mathbf{10 \times 20}$
Rel. pose 8pt one-sided rad. dist.	Kuang et al.[30]	12×24	$\mathbf{11 \times 20}$	$\mathbf{11 \times 20}$
TDOA offset rank 2, 7,4 pts	Kuang et al.[28]	$\mathbf{20 \times 15}$	$\mathbf{20 \times 15}$	$\mathbf{20 \times 15}$
Rel. pose + one focal 6pt	Bujnak et al.[4] (*)	$\mathbf{21 \times 30}$	$\mathbf{21 \times 30}$	$\mathbf{21 \times 30}$
P3.5P + focal	Wu [52]	$\mathbf{20 \times 43}$	24×45	20×44
Rel. pose + const. focal 6pt	Kukelova et al.[33] (*)	$\mathbf{31 \times 46}$	31×50	31×50
Rel. pose + rad. dist. 8pt	Kukelova et al.[33] (*)	32×48	$\mathbf{31 \times 49}$	32×50
Rel. pose 6pt ones-sided rad. dist.	Kuang et al.[30]	48×70	$\mathbf{34 \times 60}$	$\mathbf{34 \times 60}$
TDOA offset rank 2, 5,6 pts	Kuang et al.[28]	105×83	105×83	$\mathbf{40 \times 42}$
Rolling shutter pose	Saurer et al.[44] (*)	48×56	50×55	$\mathbf{47 \times 55}$
Generalized P4P + scale	Ventura et al.[51] (*)	48×56	50×55	$\mathbf{47 \times 55}$
Stitching + const. focal + rad. dist. 3pt	Naroditsky et al.[39]	54×77	96×108	$\mathbf{48 \times 66}$
TDOA offset rank 3, 9,5 pts	Kuang et al.[28]	$\mathbf{70 \times 31}$	$\mathbf{70 \times 31}$	$\mathbf{70 \times 31}$
TDOA offset rank 3, 7,6 pts	Kuang et al.[28]	255×157	255×157	$\mathbf{75 \times 57}$
Generalized rel. pose 6pt	Stewénius et al.[48]	$\mathbf{60 \times 120}^\ddagger$	135×164	99×163
Optimal PnP	Hesch et al.[21]	120×120	93×116	$\mathbf{88 \times 115}$
Triangulation from satellite im.	Zheng et al.[53] (*)	93×120	93×116	$\mathbf{88 \times 115}$
Optimal PnP (Cayley)	Nakano [38] (*)	124×164	186×161	$\mathbf{118 \times 158}$
P4P + focal + rad. dist.	Bujnak et al.[5] (*)	136×152	$\mathbf{140 \times 144}$	140×156
Rel. pose + rad. dist. 6pt	Kukelova et al.[33] (*)	238×290	223×290	$\mathbf{154 \times 210}$
Rel. pose + 2 rad. dist. 9pt	Kukelova et al.[33] (*)	179×203	355×298	$\mathbf{165 \times 200}$
Rel. pose 7pt one-sided focal + rad. dist.	Kuang et al.[30]	200×231	249×214	$\mathbf{185 \times 204}$
Weak PnP	Larsson et al.[35]	234×276	$568 \times 498^\dagger$	$\mathbf{189 \times 232}^\dagger$
Weak PnP (2x2 sym)	Larsson et al.[35]	104×90	$83 \times 90^\dagger$	$\mathbf{49 \times 59}^\dagger$
Rolling shutter R6P	Albl et al.[2] (*)	$\mathbf{196 \times 216}$	222×230	204×224
Optimal pose w dir 4pt	Svärm et al.[49]	280×252	371×351	$\mathbf{203 \times 239}$
Rel. pose w dir. 3pt	Saurer et al.[45] (*)	411×489	287×324	$\mathbf{210 \times 255}$
Rel. pose w dir. 3pt (using sym.)	-	-	$94 \times 111^\dagger$	$\mathbf{40 \times 57}^\dagger$
Abs. pose quivers	Kuang et al.[27]	372×386	420×406	$\mathbf{217 \times 253}$
L_2 3 view triangulation (Relaxed)	Kukelova et al.[34] (*)	274×305	399×384	$\mathbf{239 \times 290}$
Rel. pose w angle 4pt	Li et al.[36] (*)	$\mathbf{270 \times 290}$	280×304	266×329
Refractive P5P	Haner et al.[16]	280×399	410×480	$\mathbf{240 \times 324}$
TDOA offset rank 3, 6,8 pts	Kuang et al.[28]	1359×754	1359×754	$\mathbf{356 \times 345}$
Optimal PnP	Zheng et al.[54] (*)	575×656	812×704	$\mathbf{521 \times 601}$
Optimal PnP (using sym.)	Zheng et al.[54] (*)	348×376	$484 \times 408^\dagger$	$\mathbf{302 \times 342}^\dagger$
Optimal pose w dir 3pt	Svärm et al.[49]	$1,260 \times 1,278$	918×726	$\mathbf{544 \times 592}$
Optimal PnP (quaternion)	Nakano [38] (*)	630×710	958×693	$\mathbf{604 \times 684}$
Refractive P6P + focal	Haner et al.[16]	648×917	$2,196 \times 1,913^\dagger$	$\mathbf{636 \times 851}^\dagger$
Rel. pose + const. focal + rad. dist. 7pt	Jiang et al.[23]	$886 \times 1,011$	$1,393 \times 1,237$	$\mathbf{581 \times 862}$
Dual-Receiver TDOA 5pt	Burgess et al.[7]	$2,625 \times 2,352$	$850 \times 1,167$	$\mathbf{455 \times 768}$
Optimal PnP (rot. matrix)	Nakano [38] (*)	$1,936 \times 1,976$	$1,698 \times 1,153$	$\mathbf{1,102 \times 1,135}$
L_2 3 view triangulation	Kukelova et al.[34] (*)	$1,866 \times 1,975$	$2,647 \times 2,584$	$\mathbf{1,759 \times 2,013}$

(*) Original template constructed using [33]. If several elimination templates are used, the largest of these templates is reported.
†: The problem contains variable-aligned symmetries [3, 31, 35] that was automatically found and removed by our generator.
‡: The original template doesn't generate the full Gröbner basis, and some additional operations on the template are performed.