5.4. Derivation of $\bar{f}_{ij}$

$$
\begin{aligned}
\bar{f}_{i1} &\triangleq \mathbf{u}_i^T\lfloor\mathbf{k}_1\rfloor^2\lfloor\mathbf{k}_3''\rfloor^2\mathbf{v}_i'' \\
&= \mathbf{u}_i^T\lfloor\mathbf{k}_1\rfloor(\mathbf{k}_3''\mathbf{k}_1^T - (\mathbf{k}_1^T\mathbf{k}_3'')\mathbf{I})\lfloor\mathbf{k}_3''\rfloor\mathbf{v}_i'' \\
&= (\mathbf{u}_i^T\lfloor\mathbf{k}_1\rfloor\mathbf{k}_3'')(\mathbf{k}_1^T\lfloor\mathbf{k}_3''\rfloor\mathbf{v}_i'') \\
&= ((\mathbf{u}_i\times\mathbf{k}_1)^T\mathbf{k}_3'')(\mathbf{k}_1^T\mathbf{C}(\mathbf{k}_1,\phi)\mathbf{C}(\mathbf{k}_2,\theta_2)\lfloor\mathbf{k}_3\rfloor\mathbf{v}_i) \\
&= \delta\mathbf{k}_1'^{\,T}\lfloor\mathbf{k}_3\rfloor\mathbf{v}_i \\
&= \delta\mathbf{v}_i^T\mathbf{k}_2 \\
\bar{f}_{i2} &\triangleq \mathbf{u}_i^T\lfloor\mathbf{k}_1\rfloor^2\lfloor\mathbf{k}_3''\rfloor\mathbf{v}_i'' \\
&= (\mathbf{u}_i^T\lfloor\mathbf{k}_1\rfloor\mathbf{k}_3'')(\mathbf{k}_1^T\mathbf{v}_i'') \\
&= \delta(\mathbf{k}_1^T\mathbf{C}(\mathbf{k}_1,\phi)\mathbf{C}(\mathbf{k}_2,\theta_2)\mathbf{v}_i) \\
&= \delta\mathbf{v}_i^T\mathbf{k}_1' \\
\bar{f}_{i3} &\triangleq (\mathbf{k}_3''^T\mathbf{v}_i'')\mathbf{u}_i^T\lfloor\mathbf{k}_1\rfloor\mathbf{k}_3'' \\
&= \delta\mathbf{k}_3^T\mathbf{C}(\mathbf{k}_2,-\theta_2)\mathbf{C}(\mathbf{k}_1,-\phi)\mathbf{C}(\mathbf{k}_1,\phi)\mathbf{C}(\mathbf{k}_2,\theta_2)\mathbf{v}_i \\
&= \delta\mathbf{v}_i^T\mathbf{k}_3 \\
\bar{f}_{i4} &\triangleq -(\mathbf{u}_i^T\mathbf{k}_1)\mathbf{k}_1^T\lfloor\mathbf{k}_3''\rfloor^2\mathbf{v}_i'' \\
&= -(\mathbf{u}_i^T\mathbf{k}_1)\mathbf{k}_1(\mathbf{k}_3''\mathbf{k}_3''^T - \mathbf{I})\mathbf{v}_i'' \\
&= (\mathbf{u}_i^T\mathbf{k}_1)(\mathbf{k}_1\mathbf{v}_i'') \\
&= (\mathbf{u}_i^T\mathbf{k}_1)(\mathbf{v}_i^T\mathbf{k}_1') \\
\bar{f}_{i5} &\triangleq -(\mathbf{u}_i^T\mathbf{k}_1)\mathbf{k}_1^T\lfloor\mathbf{k}_3''\rfloor\mathbf{v}_i'' = -(\mathbf{u}_i^T\mathbf{k}_1)(\mathbf{v}_i^T\mathbf{k}_2)
\end{aligned}
$$

5.5. Derivation of f_{ij} and $\bar{\bar{\mathbf{C}}}$

First, note that

$$
\begin{aligned}
\mathbf{v}_i^T\mathbf{k}_2 &= ({}^C\mathbf{b}_i\times{}^C\mathbf{b}_3)^T(\mathbf{k}_1'\times\mathbf{k}_3) \\
&= ({}^C\mathbf{b}_i^T\mathbf{k}_1')({}^C\mathbf{b}_3^T\mathbf{k}_3) - ({}^C\mathbf{b}_i^T\mathbf{k}_3)({}^C\mathbf{b}_3^T\mathbf{k}_1') \\
&= ({}^C\mathbf{b}_i^T\mathbf{k}_1')({}^C\mathbf{b}_3^T\mathbf{k}_3) \\
\mathbf{v}_i^T\mathbf{k}_1' &= -({}^C\mathbf{b}_i\times{}^C\mathbf{b}_3)^T(\mathbf{k}_2\times\mathbf{k}_3) \\
&= -({}^C\mathbf{b}_i^T\mathbf{k}_2)({}^C\mathbf{b}_3^T\mathbf{k}_3) + ({}^C\mathbf{b}_i^T\mathbf{k}_3)({}^C\mathbf{b}_3^T\mathbf{k}_2) \\
&= -({}^C\mathbf{b}_i^T\mathbf{k}_2)({}^C\mathbf{b}_3^T\mathbf{k}_3)
\end{aligned}
$$

Let $\psi \triangleq \theta_3 - \theta_3'$, and thus

$$
\begin{bmatrix}\cos\theta_3' \\ \sin\theta_3'\end{bmatrix} = \begin{bmatrix}\cos\psi\cos\theta_3 + \sin\psi\sin\theta_3 \\ \cos\psi\cos\theta_3 + \sin\psi\sin\theta_3\end{bmatrix} \tag{64}
$$

From (64) and (27), we get

$$
\begin{aligned}
\cos\psi &= \frac{\bar{f}_{11}}{\sqrt{\bar{f}_{11}^2+\bar{f}_{12}^2}} = -\frac{\bar{f}_{15}}{\sqrt{\bar{f}_{14}^2+\bar{f}_{15}^2}} \\
&= \frac{\mathbf{v}_1^T\mathbf{k}_2}{\sqrt{(\mathbf{v}_1^T\mathbf{k}_2)^2+(\mathbf{v}_1^T\mathbf{k}_1')^2}} \\
&= \frac{{}^C\mathbf{b}_1^T\mathbf{k}_1'}{\sqrt{({}^C\mathbf{b}_1^T\mathbf{k}_2)^2+({}^C\mathbf{b}_1^T\mathbf{k}_1')^2}} \\
&= \frac{{}^C\mathbf{b}_1^T\mathbf{k}_1'}{\|\mathbf{k}_3\times{}^C\mathbf{b}_1\|} \\
&= {}^C\mathbf{b}_1^T\mathbf{k}_1'
\end{aligned}
$$

$$
\begin{aligned}
\sin\psi &= \frac{\bar{f}_{12}}{\sqrt{\bar{f}_{11}^2+\bar{f}_{12}^2}} = \frac{\bar{f}_{14}}{\sqrt{\bar{f}_{14}^2+\bar{f}_{15}^2}} \\
&= \frac{\mathbf{v}_1^T\mathbf{k}_1'}{\sqrt{(\mathbf{v}_1^T\mathbf{k}_2)^2+(\mathbf{v}_1^T\mathbf{k}_1')^2}} \\
&= -{}^C\mathbf{b}_1^T\mathbf{k}_2
\end{aligned}
$$

Then, from (26) and (28), we derive the expressions of f_{ij}:

$$
\begin{aligned}
f_{11} &= \bar{f}_{11}\cos\psi + \bar{f}_{12}\sin\psi \\
&= \delta({}^C\mathbf{b}_3^T\mathbf{k}_3)(({}^C\mathbf{b}_1^T\mathbf{k}_2)^2 + ({}^C\mathbf{b}_1^T\mathbf{k}_1')^2) \\
&= \delta({}^C\mathbf{b}_3^T\mathbf{k}_3) \\
f_{21} &= \bar{f}_{21}\cos\psi + \bar{f}_{22}\sin\psi \\
&= \delta({}^C\mathbf{b}_3^T\mathbf{k}_3)(({}^C\mathbf{b}_2^T\mathbf{k}_2)({}^C\mathbf{b}_1^T\mathbf{k}_2) + ({}^C\mathbf{b}_2^T\mathbf{k}_1')({}^C\mathbf{b}_1^T\mathbf{k}_1')) \\
&= \delta({}^C\mathbf{b}_3^T\mathbf{k}_3)({}^C\mathbf{b}_2^T(\mathbf{k}_2\mathbf{k}_2^T + \mathbf{k}_1'\mathbf{k}_1'^{\,T}){}^C\mathbf{b}_1) \\
&= \delta({}^C\mathbf{b}_3^T\mathbf{k}_3)({}^C\mathbf{b}_2^T(\mathbf{I} - \mathbf{k}_3\mathbf{k}_3^T){}^C\mathbf{b}_1) \\
&= \delta({}^C\mathbf{b}_3^T\mathbf{k}_3)({}^C\mathbf{b}_2^{T\,C}\mathbf{b}_1) \\
f_{22} &= -\bar{f}_{21}\sin\psi + \bar{f}_{22}\cos\psi \\
&= \delta({}^C\mathbf{b}_3^T\mathbf{k}_3)(({}^C\mathbf{b}_2^T\mathbf{k}_1')({}^C\mathbf{b}_1^T\mathbf{k}_2) - ({}^C\mathbf{b}_2^T\mathbf{k}_2)({}^C\mathbf{b}_1^T\mathbf{k}_1')) \\
&= \delta({}^C\mathbf{b}_3^T\mathbf{k}_3)({}^C\mathbf{b}_2^T(\mathbf{k}_2\mathbf{k}_2^T + \mathbf{k}_1'\mathbf{k}_1'^{\,T}){}^C\mathbf{b}_1) \\
&= \delta({}^C\mathbf{b}_3^T\mathbf{k}_3)({}^C\mathbf{b}_2\times{}^C\mathbf{b}_1)^T(\mathbf{k}_1'\times\mathbf{k}_2) \\
&= \delta({}^C\mathbf{b}_3^T\mathbf{k}_3)\|{}^C\mathbf{b}_2\times{}^C\mathbf{b}_1\|\mathbf{k}_3^T\mathbf{k}_3 \\
&= \delta({}^C\mathbf{b}_3^T\mathbf{k}_3)\|{}^C\mathbf{b}_2\times{}^C\mathbf{b}_1\| \\
f_{15} &= \bar{f}_{15}\cos\psi - \bar{f}_{14}\sin\psi \\
&= -(\mathbf{u}_1^T\mathbf{k}_1)f_{11}/\delta \\
&= -(\mathbf{u}_1^T\mathbf{k}_1)({}^C\mathbf{b}_3^T\mathbf{k}_3) \\
f_{24} &= \bar{f}_{25}\sin\psi + \bar{f}_{24}\cos\psi \\
&= (\mathbf{u}_2^T\mathbf{k}_1)f_{22}/\delta \\
&= (\mathbf{u}_2^T\mathbf{k}_1)({}^C\mathbf{b}_3^T\mathbf{k}_3)\|{}^C\mathbf{b}_2\times{}^C\mathbf{b}_1\| \\
f_{25} &= \bar{f}_{25}\cos\psi - \bar{f}_{24}\sin\psi \\
&= -(\mathbf{u}_2^T\mathbf{k}_1)f_{21}/\delta \\
&= -(\mathbf{u}_2^T\mathbf{k}_1)({}^C\mathbf{b}_3^T\mathbf{k}_3)({}^C\mathbf{b}_1^T\mathbf{k}_1')
\end{aligned}
$$

Additionally, we can derive the expression of $\bar{\bar{\mathbf{C}}}$, which is defined in (56):

$$
\begin{aligned}
\bar{\bar{\mathbf{C}}} &= \mathbf{C}(\mathbf{e}_2,\theta_3-\theta_3')\bar{\mathbf{C}}^T\mathbf{C}(\mathbf{k}_1,\phi)\mathbf{C}(\mathbf{k}_2,\theta_2) \\
&= \mathbf{C}(\mathbf{e}_2,\psi)\begin{bmatrix}\mathbf{k}_1 & \mathbf{k}_3' & \mathbf{k}_2\end{bmatrix}^T\mathbf{C}(\mathbf{k}_2,\theta_2) \\
&= \mathbf{C}(\mathbf{e}_2,\psi)\begin{bmatrix}\mathbf{k}_1' & \mathbf{k}_3 & \mathbf{k}_2\end{bmatrix}^T \\
&= \begin{bmatrix}\cos\psi\mathbf{k}_1' - \sin\psi\mathbf{k}_2 & \mathbf{k}_3 & \sin\psi\mathbf{k}_1' + \cos\psi\mathbf{k}_2\end{bmatrix}^T \\
&= \begin{bmatrix}(\mathbf{k}_1'\mathbf{k}_1'^{\,T} + \mathbf{k}_2\mathbf{k}_2^T){}^C\mathbf{b}_1 & \mathbf{k}_3 & \sin\psi\mathbf{k}_1' + \cos\psi\mathbf{k}_2\end{bmatrix}^T \\
&= \begin{bmatrix}(\mathbf{I} - \mathbf{k}_3\mathbf{k}_3^T){}^C\mathbf{b}_1 & \mathbf{k}_3 & \sin\psi\mathbf{k}_1' + \cos\psi\mathbf{k}_2\end{bmatrix}^T \\
&= \begin{bmatrix}{}^C\mathbf{b}_1 & \mathbf{k}_3 & \sin\psi\mathbf{k}_1' + \cos\psi\mathbf{k}_2\end{bmatrix}^T \\
&= \begin{bmatrix}{}^C\mathbf{b}_1 & \mathbf{k}_3 & {}^C\mathbf{b}_1\times\mathbf{k}_3\end{bmatrix}^T
\end{aligned}
$$