Supplementary for Discretely Coding Semantic Rank Orders for Supervised Image Hashing

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Theorem 1. *Let $\{\mathbf{B}^{(t)}\}$ be generated by the update rule of Algorithm 1, then $\{f(\mathbf{B}^{(t)})\}$ is monotonically non-increasing, i.e., $f(\mathbf{B}^{(t+1)}) \leq f(\mathbf{B}^{(t)})$, and $\{\mathbf{B}^{(t)}\}$ converges.*

Proof. According to the linearization $\hat{f}_t(\mathbf{B}) = f(\mathbf{B}^{(t)}) + \langle \nabla f(\mathbf{B}^{(t)}), \mathbf{B} - \mathbf{B}^{(t)} \rangle$, we can observe that $\hat{f}_t(\mathbf{B}^{(t)}) = f(\mathbf{B}^{(t)})$ and we have

$$\begin{aligned}
&\hat{f}_t(\mathbf{B}^{(t+1)}) - \hat{f}_t(\mathbf{B}^{(t)}) = \langle \nabla f(\mathbf{B}^{(t)}), \mathbf{B}^{(t+1)} - \mathbf{B}^{(t)} \rangle \\
&= \sum_{p,q} \left\langle \nabla_{p,q} f(\mathbf{B}^{(t)}), \left(\mathbf{B}^{(t+1)}\right)_{p,q} - \left(\mathbf{B}^{(t)}\right)_{p,q} \right\rangle.
\end{aligned}$$

For each term in the above summation, $\left\langle \nabla_{p,q} f(\mathbf{B}^{(t)}), \left(\mathbf{B}^{(t+1)}\right)_{p,q} - \left(\mathbf{B}^{(t)}\right)_{p,q} \right\rangle \neq 0$ if and only if $\nabla_{p,q} f(\mathbf{B}^{(t)}) \neq 0$ and $\left(\mathbf{B}^{(t+1)}\right)_{p,q} - \left(\mathbf{B}^{(t)}\right)_{p,q} \neq 0$.

There are several cases for $\nabla_{p,q} f(\mathbf{B}^{(t)})$ and $\left(\mathbf{B}^{(t+1)}\right)_{p,q} - \left(\mathbf{B}^{(t)}\right)_{p,q}$. If $\nabla_{p,q} f(\mathbf{B}^{(t)}) = 0$ for all p, q, then $f(\mathbf{B}^{(t)})$ is the local minimum value and there is no further update iteration.

Let us consider the case that there exist p, q such that $\nabla_{p,q} f(\mathbf{B}^{(t)}) \neq 0$. If $\left(\mathbf{B}^{(t+1)}\right)_{p,q} - \left(\mathbf{B}^{(t)}\right)_{p,q} = 0$, then $\langle \nabla_{p,q} f(\mathbf{B}^{(t)}), \left(\mathbf{B}^{(t+1)}\right)_{p,q} - \left(\mathbf{B}^{(t)}\right)_{p,q} \rangle = 0$. If $\left(\mathbf{B}^{(t+1)}\right)_{p,q} - \left(\mathbf{B}^{(t)}\right)_{p,q} \neq 0$, i.e., $\left(\mathbf{B}^{(t+1)}\right)_{p,q} = -\left(\mathbf{B}^{(t)}\right)_{p,q}$, according to the update rule,

$$\left(\mathbf{B}^{(t+1)}\right)_{p,q} = \operatorname{sign}(-\nabla_{p,q} f(\mathbf{B}^{(t)})).$$

In this case,

$$\begin{aligned}
&\langle \nabla_{p,q} f(\mathbf{B}^{(t)}), \left(\mathbf{B}^{(t+1)}\right)_{p,q} - \left(\mathbf{B}^{(t)}\right)_{p,q} \rangle \\
&\quad = \langle \nabla_{p,q} f(\mathbf{B}^{(t)}), \left(\mathbf{B}^{(t+1)}\right)_{p,q} + \left(\mathbf{B}^{(t+1)}\right)_{p,q} \rangle \\
&\quad = \langle \nabla_{p,q} f(\mathbf{B}^{(t)}), 2\operatorname{sign}(-\nabla_{p,q} f(\mathbf{B}^{(t)})) \rangle \\
&\quad < 0.
\end{aligned} \tag{1}$$

Therefore, if $f(\mathbf{B}^{(t)})$ is not a local minimum value, we have

$$\hat{f}_t(\mathbf{B}^{(t+1)}) - \hat{f}_t(\mathbf{B}^{(t)}) = \langle \nabla f(\mathbf{B}^{(t)}), \mathbf{B}^{(t+1)} - \mathbf{B}^{(t)} \rangle < 0$$

which means $\hat{f}_t(\mathbf{B}^{(t+1)}) < \hat{f}_t(\mathbf{B}^{(t)})$ for any variable $S \neq \emptyset$ in the update rule of $\mathbf{B}^{(t)}$.

If $f(\mathbf{B}^{(t+1)})$ is already less than $f(\mathbf{B}^{(t)})$, then f is monotonically decreasing by the update rule. If $f(\mathbf{B}^{(t+1)}) > f(\mathbf{B}^{(t)})$, then we prove that there exists S such that $\mathbf{B}^{(t+1)} = \mathcal{F}(\operatorname{sign}(-\nabla f(\mathbf{B}^{(t)})), \mathbf{B}^{(t)}, S)$ satisfies $f(\mathbf{B}^{(t+1)}) \leq f(\mathbf{B}^{(t)})$, where matrix function $\mathcal{F}(\mathbf{M}, \mathbf{N}, S)$ is defined as

$$\left(\mathcal{F}(\mathbf{M}, \mathbf{N}, S)\right)_{p,q} = \begin{cases} \mathbf{M}_{p,q} & \text{if } (p,q) \in S, \\ \mathbf{N}_{p,q} & \text{otherwise}, \end{cases}$$

with $S \subseteq \{(p,q) | \mathbf{M}_{p,q} \neq \mathbf{N}_{p,q}, \mathbf{M}_{p,q} \neq 0\}$ and the size of S is controlled by a parameter φ. Then we have

$$\begin{aligned}
f(\mathbf{B}^{(t)}) &= \hat{f}_t(\mathbf{B}^{(t)}) = \hat{f}_t(\mathcal{F}(\operatorname{sign}(-\nabla f(\mathbf{B}^{(t)})), \mathbf{B}^{(t)}, \emptyset)) \\
&> \hat{f}_t(\mathcal{F}(\operatorname{sign}(-\nabla f(\mathbf{B}^{(t)})), \mathbf{B}^{(t)}, S)) = \hat{f}_t(\mathbf{B}^{(t+1)})
\end{aligned}$$

for any $S \neq \emptyset$. In this way, we try to find a small neighborhood of $\mathbf{B}^{(t)}$, on which $\hat{f}_t(\mathbf{B})$ can approximately represent $f(\mathbf{B})$. Considering the variable S with $|S| = 1$ (i.e.,

$\varphi = \frac{1}{nm}$), the following set:

$$\mathcal{N}_1 = \Big\{ \mathcal{F}(\mathrm{sign}(-\nabla f(\mathbf{B}^{(t)})), \mathbf{B}^{(t)}, S) \Big| \\ S \subseteq \big\{ (p,q) | \big(\mathrm{sign}(-\nabla f(\mathbf{B}^{(t)}))\big)_{p,q} \neq \big(\mathbf{B}^{(t)}\big)_{p,q} \\ \text{and } \big(\mathrm{sign}(-\nabla f(\mathbf{B}^{(t)}))\big)_{p,q} \neq 0 \big\} \Big\}$$

contains all the neighbor binary points of $\mathbf{B}^{(t)}$ in the descending direction.

If $\exists\ \mathbf{B}_1 \in \mathcal{N}_1$ such that $f(\mathbf{B}_1) \leq \hat{f}_t(\mathbf{B}_1)$, we have $f(\mathbf{B}_1) \leq \hat{f}_t(\mathbf{B}_1) < \hat{f}_t(\mathbf{B}^{(t)}) = f(\mathbf{B}^{(t)})$. Then there exists S such that $f(\mathbf{B}^{(t+1)}) \leq f(\mathbf{B}^{(t)})$.

If $\exists\ \mathbf{B}_1 \in \mathcal{N}_1$ such that $|f(\mathbf{B}_1) - \hat{f}_t(\mathbf{B}_1)| \leq \hat{f}_t(\mathbf{B}^{(t)}) - \hat{f}_t(\mathbf{B}_1)$, we have $f(\mathbf{B}_1) - \hat{f}_t(\mathbf{B}_1) \leq \hat{f}_t(\mathbf{B}^{(t)}) - \hat{f}_t(\mathbf{B}_1)$, which leads to $f(\mathbf{B}_1) \leq \hat{f}_t(\mathbf{B}^{(t)}) = f(\mathbf{B}^{(t)})$. Then there exists S such that $f(\mathbf{B}^{(t+1)}) \leq f(\mathbf{B}^{(t)})$.

If $\forall\ \mathbf{B} \in \mathcal{N}_1$ we have $f(\mathbf{B}) > \hat{f}_t(\mathbf{B})$ and $|f(\mathbf{B}) - \hat{f}_t(\mathbf{B})| > \hat{f}_t(\mathbf{B}^{(t)}) - \hat{f}_t(\mathbf{B})$, then $f(\mathbf{B}) > \hat{f}_t(\mathbf{B}^{(t)}) = f(\mathbf{B}^{(t)})$, $\forall\ \mathbf{B} \in \mathcal{N}_1$. In this condition, the function value on every neighbor binary point of $\mathbf{B}^{(t)}$ in the descending direction is larger than $f(\mathbf{B}^{(t)})$, which renders $f(\mathbf{B}^{(t)})$ is a local minimum value for the domain $\{\pm 1\}^{m\times n}$.

Therefore, $\{f(\mathbf{B}^{(t)})\}$ is monotonically non-increasing. Since $\{\pm 1\}^{m\times n}$ is a finite set and f is a bounded function ($f \geq 0$), $\{\mathbf{B}^{(t)}\}$ will converge under the above-defined update rule. $\square$