Supplementary Material
Revisiting Metric Learning for SPD Matrix based Visual Representation

Luping Zhou, Lei Wang
University of Wollongong, Australia
lupingz,leiw@uow.edu.au

Jianjia Zhang
CSIRO, Australia
seuzjj@gmail.com

Yinghuan Shi, Yang Gao
Nanjing University, China
syh,gaoy@nju.edu.cn

1. Proof of Proposition in Section 3.5.2

Proposition. Given a SPD matrix $\mathbf{X}_{d\times d}$, let $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_d]$ and $\mathbf{D} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_d)$ denote the eigenvector and eigenvalue matrices in full, respectively. Let $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \cdots, \alpha_d]^\top$ be the power of the eigenvalues of $\mathbf{X}$, i.e., $\mathbf{X}(\boldsymbol{\alpha}) = \mathbf{U}\mathbf{D}^{\boldsymbol{\alpha}}\mathbf{U}^\top$, and $\mathrm{vec}(\cdot)$ denote the vectorisation of a matrix. It can be shown that $\mathrm{vec}(\log(\mathbf{X}(\boldsymbol{\alpha}))) = \boldsymbol{\Gamma}\boldsymbol{\alpha}$, where $\boldsymbol{\Gamma} = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_d]$ and $\mathbf{v}_i \equiv \mathrm{vec}((\log\lambda_i)\mathbf{u}_i\mathbf{u}_i^\top)$, $\mathbf{v}_i \in \mathbb{R}^{d^2}$. The columns of $\boldsymbol{\Gamma}$ form a set of d orthogonal bases spanning a d-dimensional subspace $\mathcal{V}$ in the whole space of $\mathbb{R}^{d^2}$.

Proof. By eigen-decomposition, we have $\mathbf{X} = \mathbf{U}\mathbf{D}\mathbf{U}^\top = \lambda_1\mathbf{u}_1\mathbf{u}_1^\top + \lambda_2\mathbf{u}_2\mathbf{u}_2^\top + \cdots + \lambda_d\mathbf{u}_d\mathbf{u}_d^\top$. It is easy to obtain that $\log(\mathbf{X}) = \mathbf{U}\log[\mathbf{D}]\mathbf{U}^\top = \sum_{i=1}^d(\log\lambda_i)\mathbf{u}_i\mathbf{u}_i^\top$ and $\log(\mathbf{X}(\boldsymbol{\alpha})) = \sum_{i=1}^d(\alpha_i\log\lambda_i)\mathbf{u}_i\mathbf{u}_i^\top$, where $\log[\mathbf{D}]$ denotes the diagonal matrix obtained after applying the natural logarithm to the diagonal elements of $\mathbf{D}$ (The square bracket $[\cdot]$ is used to differentiate it from the matrix logarithm). Therefore, it holds that

$$\begin{aligned}\mathrm{vec}(\log(\mathbf{X}(\boldsymbol{\alpha}))) &= \mathrm{vec}\left((\alpha_1\log\lambda_1)\mathbf{u}_1\mathbf{u}_1^\top\right) + \cdots + \mathrm{vec}\left((\alpha_d\log\lambda_d)\mathbf{u}_d\mathbf{u}_d^\top\right) \\ &= \alpha_1\mathbf{v}_1 + \cdots + \alpha_d\mathbf{v}_d \\ &= \boldsymbol{\Gamma}\boldsymbol{\alpha}.\end{aligned} \tag{1}$$

Now let's prove that $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_d\}$ forms a set of d orthogonal bases for a d-dimensional subspace (denoted by $\mathcal{V}$) in $\mathbb{R}^{d^2}$, that is, $\langle\mathbf{v}_i, \mathbf{v}_j\rangle = 0$, for all $1 < i < j < d$.

It can be shown that

$$\begin{aligned}\langle\mathbf{v}_i, \mathbf{v}_j\rangle &= \langle\mathrm{vec}\left((\log\lambda_i)\mathbf{u}_i\mathbf{u}_i^\top\right), \mathrm{vec}\left((\log\lambda_j)\mathbf{u}_j\mathbf{u}_j^\top\right)\rangle \\ &= \langle(\log\lambda_i)\mathbf{u}_i\mathbf{u}_i^\top, (\log\lambda_j)\mathbf{u}_j\mathbf{u}_j^\top\rangle_F \\ &\quad (\because \langle\mathbf{A}, \mathbf{B}\rangle_F = \mathrm{trace}(\mathbf{A}^\top\mathbf{B})) \\ &= \mathrm{trace}((\log\lambda_i)\mathbf{u}_i\mathbf{u}_i^\top(\log\lambda_j)\mathbf{u}_j\mathbf{u}_j^\top) \\ &= (\log\lambda_i\log\lambda_j)\mathrm{trace}(\mathbf{u}_i\mathbf{u}_i^\top\mathbf{u}_j\mathbf{u}_j^\top) \\ &\quad (\because \mathbf{u}_i^\top\mathbf{u}_j = 0 \text{ as two eigenvectors}) \\ &= 0.\end{aligned} \tag{2}$$

In addition, it is trivial to show that $\|\mathbf{v}_i\|^2 = (\log\lambda_i)^2$. Therefore, $\{\mathbf{v}_1, \mathbf{v}_1, \cdots, \mathbf{v}_d\}$ forms a set of d orthogonal bases for a subspace $\mathcal{V}$ in $\mathbb{R}^{d^2}$. □

2. Fifteen most difficult texture pairs in Brodatz data set used in Section 4.1

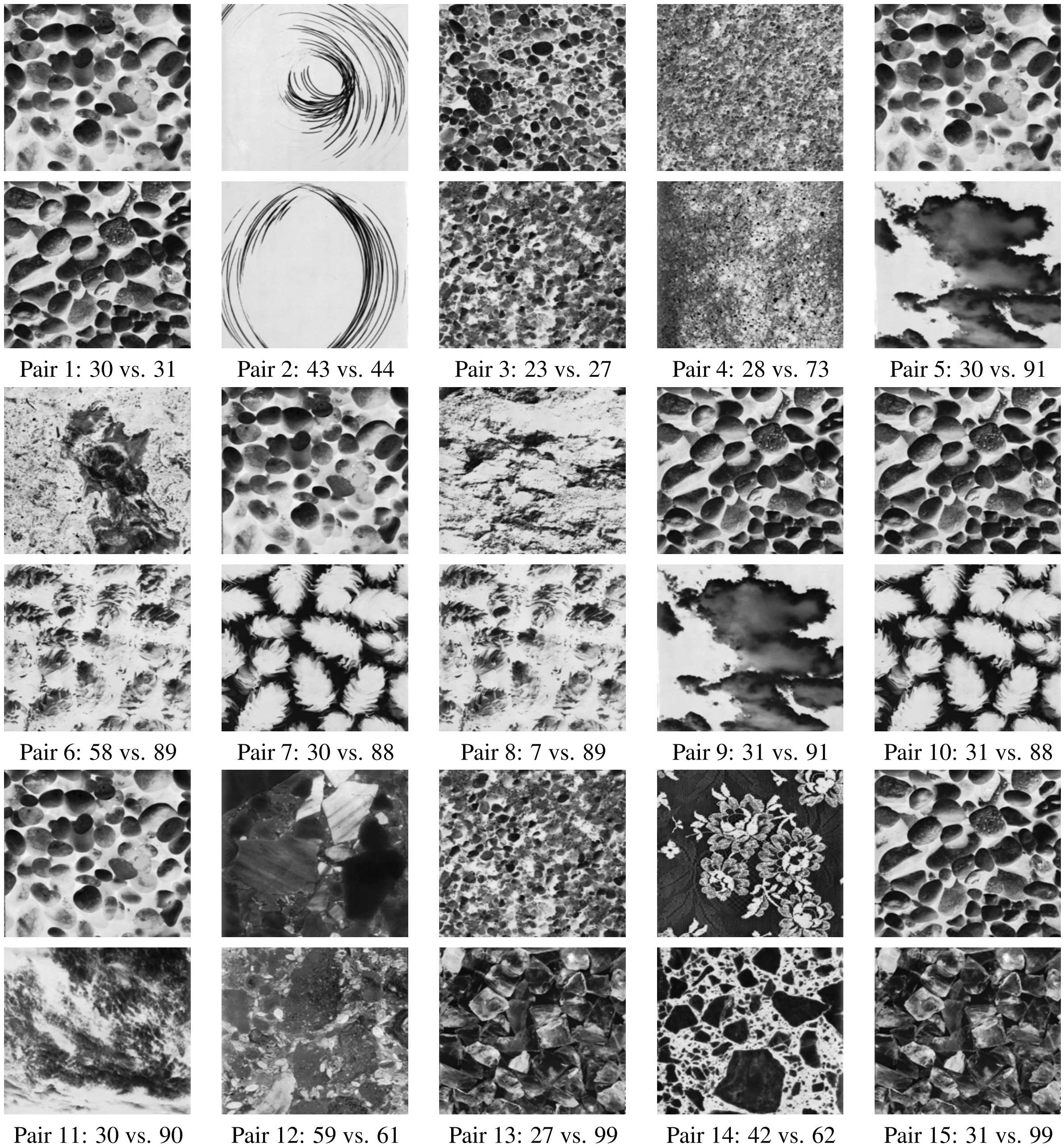

Figure 1. Fifteen most difficult texture pairs (with class labels) used in the binary classification experiment from Brodatz data set in Section 4.1.