Supplementary Material

for submission "A Fast Resection-Intersection Method for the Known Rotation Problem"

Anonymous CVPR submission

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1. Converting (Int_k) and (Res_j) into the form (1)

Firstly, recall that in the paper, the common form of the sub-problem is

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^3} \quad & \max_i \ r_i(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{c}_i^T\mathbf{x} + d_i > 0 \ \ \forall i, \end{aligned} \tag{1}$$

where $\mathbf{x}$ are the variables of interest (3D coordinates or camera position—both 3D quantities in each sub-problem):

$$r_i(\mathbf{x}) = \frac{\|\mathbf{A}_i\mathbf{x} + \mathbf{b}_i\|_p}{\mathbf{c}_i^T\mathbf{x} + d_i} \tag{2}$$

is the i-th pseudo-convex residual function, and

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{a}_{i,1}^T \\ \mathbf{a}_{i,2}^T \end{bmatrix} \in \mathbb{R}^{2\times 3}, \ \ \mathbf{b}_i = \begin{bmatrix} b_{i,1} \\ b_{i,2} \end{bmatrix} \in \mathbb{R}^2, \ \ \mathbf{c}_i \in \mathbb{R}^3 \ \ \text{and} \ \ d_i \in \mathbb{R} \tag{3}$$

are constants.

1.1. From (Int_k) to (1)

In the paper, the intersection sub-problem (Int_k) is defined as

$$\begin{aligned} \min_{\mathbf{s}_k} \quad & \max_j \left\| \mathbf{u}_{j,k} - \frac{\mathbf{R}_j^{1:2}\mathbf{s}_k + \mathbf{t}_j^{1:2}}{\mathbf{R}_j^3\mathbf{s}_k + \mathbf{t}_j^3} \right\|_p \\ \text{s.t.} \quad & \mathbf{R}_j^3\mathbf{s}_k + \mathbf{t}_j^3 > 0 \ \ \forall j, \end{aligned} \tag{Int_k}$$

where

$$\mathbf{u}_{j,k} = \begin{bmatrix} \mathbf{u}_{j,k}^1 \\ \mathbf{u}_{j,k}^2 \end{bmatrix} \in \mathbb{R}^2 \tag{4}$$

is the 2D reprojection of the k^{th} scene point $\mathbf{s}_k$ onto the j^{th} image. Then, defining the following constants transfers (Int_k) into the form of (1).

$$\begin{aligned} \mathbf{x} &= \mathbf{s}_k, \\ \mathbf{a}_{i,1}^T &= \mathbf{u}_{j,k}^1\mathbf{R}_j^3 - \mathbf{R}_j^1, \\ \mathbf{a}_{i,2}^T &= \mathbf{u}_{j,k}^2\mathbf{R}_j^3 - \mathbf{R}_j^2, \\ b_{i,1} &= \mathbf{u}_{j,k}^1\mathbf{t}_j^3 - \mathbf{t}_j^1, \\ b_{i,2} &= \mathbf{u}_{j,k}^2\mathbf{t}_j^3 - \mathbf{t}_j^2, \\ \mathbf{c}_i^T &= \mathbf{R}_j^3\mathbf{s}_k + \mathbf{t}_j^3, \ \ \text{and} \\ d_i &= \mathbf{t}_j^3. \end{aligned} \tag{5}$$

1.2. From (Res_j) to (1)

The resection sub-problem (Res_j) is defined as

$$\begin{aligned} \min_{\mathbf{t}_j} \quad & \max_{k} \left\| \mathbf{u}_{j,k} - \frac{\mathbf{R}_j^{1:2}\mathbf{s}_k + \mathbf{t}_j^{1:2}}{\mathbf{R}_j^{3}\mathbf{s}_k + \mathbf{t}_j^{3}} \right\|_p \\ \text{s.t.} \quad & \mathbf{R}_j^{3}\mathbf{s}_k + \mathbf{t}_j^{3} > 0 \ \ \forall k, \end{aligned} \tag{Res_j}$$

where $\mathbf{u}_{j,k}$ has the same definition as in (4). Then, defining the following constants transfers (Res_j) into the form of (1).

$$\begin{aligned} \mathbf{x} =& \mathbf{t}_j, \\ \mathbf{a}_{i,1}^T =& [-1, 0, \mathbf{u}_{j,k}^1], \\ \mathbf{a}_{i,2}^T =& [0, -1, \mathbf{u}_{j,k}^2], \\ b_{i,1} =& \mathbf{u}_{j,k}^1 \mathbf{R}_j^3 \mathbf{s}_k - \mathbf{R}_j^1 \mathbf{s}_k, \\ b_{i,2} =& \mathbf{u}_{j,k}^2 \mathbf{R}_j^3 \mathbf{s}_k - \mathbf{R}_j^2 \mathbf{s}_k, \\ \mathbf{c}_i^T =& [0, 0, 1], \quad \text{and} \\ d_i =& \mathbf{R}_j^3 \mathbf{s}_k. \end{aligned} \tag{6}$$

2. Gradient formula

2.1. Gradient of residual function when $p = 2$

As seen in (2) above, if $p = 2$, the residual function becomes

$$r_i(\mathbf{x}) = \frac{\sqrt{(\mathbf{a}_{i,1}^T\mathbf{x} + b_{i,1})^2 + (\mathbf{a}_{i,2}^T\mathbf{x} + b_{i,2})^2}}{\mathbf{c}_i^T\mathbf{x} + d_i}. \tag{7}$$

Let

$$g(\mathbf{x}) = (\mathbf{a}_{i,1}^T\mathbf{x} + b_1)^2 + (\mathbf{a}_{i,2}^T\mathbf{x} + b_2)^2, \tag{8}$$

then (7) becomes

$$r_i(\mathbf{x}) = \frac{g(\mathbf{x})^{\frac{1}{2}}}{\mathbf{c}_i^T\mathbf{x} + d_i}. \tag{9}$$

Then by the quotient rule, the partial derivative of $r_i(\mathbf{x})$ w.r.t. x_j is

$$\frac{\partial r_i(\mathbf{x})}{\partial \mathbf{x}^j} = \frac{\frac{1}{2} \cdot g(\mathbf{x})^{-\frac{1}{2}} \cdot \frac{\partial g(\mathbf{x})}{\partial \mathbf{x}^j} \cdot (\mathbf{c}_i^T\mathbf{x} + d_i) - g(\mathbf{x})^{\frac{1}{2}} \mathbf{c}_i^j}{(\mathbf{c}_i^T\mathbf{x} + d_i)^2}. \tag{10}$$

By (8),

$$\frac{\partial g(\mathbf{x})}{\partial \mathbf{x}^j} = 2(\mathbf{a}_{i,1}^T\mathbf{x} + b_1)\mathbf{a}_{i,1}^j + 2(\mathbf{a}_{i,2}^T\mathbf{x} + b_2)\mathbf{a}_{i,2}^j. \tag{11}$$

Substituting (11) into (10) yields

$$\begin{aligned} \frac{\partial r_i(\mathbf{x})}{\partial \mathbf{x}^j} &= \frac{\frac{1}{2} \cdot g(\mathbf{x})^{-\frac{1}{2}} * \frac{\partial g(\mathbf{x})}{\partial \mathbf{x}^j} \cdot (\mathbf{c}_i^T\mathbf{x} + d_i) - g(\mathbf{x})^{\frac{1}{2}} \mathbf{c}_i^j}{(\mathbf{c}_i^T\mathbf{x} + d_i)^2} \\ &= \frac{\frac{1}{2} * g(\mathbf{x})^{-\frac{1}{2}} \cdot (\mathbf{c}_i^T\mathbf{x} + d_i) \cdot [2(\mathbf{a}_{i,1}^T\mathbf{x} + b_1)\mathbf{a}_{i,1}^j + 2(\mathbf{a}_{i,2}^T\mathbf{x} + b_2)\mathbf{a}_{i,2}^j] - g(\mathbf{x})^{\frac{1}{2}} \mathbf{c}_i^j}{(\mathbf{c}_i^T\mathbf{x} + d_i)^2} \\ &= \frac{g(\mathbf{x})^{-\frac{1}{2}} \cdot (\mathbf{c}_i^T\mathbf{x} + d_i) \cdot [(\mathbf{a}_{i,1}^T\mathbf{x} + b_1)\mathbf{a}_{i,1}^j + (\mathbf{a}_{i,2}^T\mathbf{x} + b_2)\mathbf{a}_{i,2}^j] - g(\mathbf{x})^{\frac{1}{2}} \mathbf{c}_i^j}{(\mathbf{c}_i^T\mathbf{x} + d_i)^2}, \end{aligned} \tag{12}$$

such that the gradient of $r_i(\mathbf{x})$ when $p = 2$ is

$$\nabla r_i(\mathbf{x}) = \begin{bmatrix} \frac{\partial r_i(\mathbf{x})}{\partial \mathbf{x}^1} \\ \frac{\partial r_i(\mathbf{x})}{\partial \mathbf{x}^2} \\ \frac{\partial r_i(\mathbf{x})}{\partial \mathbf{x}^3} \end{bmatrix} = \frac{g(\mathbf{x})^{-\frac{1}{2}} \cdot (\mathbf{c}_i^T \mathbf{x} + d_i) \cdot [(\mathbf{a}_{i,1}^T \mathbf{x} + b_1)\mathbf{a}_{i,1} + (\mathbf{a}_{i,2}^T \mathbf{x} + b_2)\mathbf{a}_{i,2}] - g(\mathbf{x})^{\frac{1}{2}} \mathbf{c}_i}{(\mathbf{c}_i^T \mathbf{x} + d_i)^2} \tag{13}$$

2.2. Gradient of residual function when $p = 1$

If $p = 1$, the residual function (2) becomes

$$r_i(\mathbf{x}) = \frac{|\mathbf{a}_{i,1}^T \mathbf{x} + b_{i,1}| + |\mathbf{a}_{i,2}^T \mathbf{x} + b_{i,2}|}{\mathbf{c}_i^T \mathbf{x} + d_i}, \tag{14}$$

which is the maximum over the following four terms:

$$\begin{aligned} &1. \frac{(\mathbf{a}_{i,1}^T \mathbf{x} + b_{i,1}) + (\mathbf{a}_{i,2}^T \mathbf{x} + b_{i,2})}{\mathbf{c}_i^T \mathbf{x} + d_i}, \\ &2. \frac{(\mathbf{a}_{i,1}^T \mathbf{x} + b_{i,1}) - (\mathbf{a}_{i,2}^T \mathbf{x} + b_{i,2})}{\mathbf{c}_i^T \mathbf{x} + d_i}, \\ &3. \frac{-(\mathbf{a}_{i,1}^T \mathbf{x} + b_{i,1}) + (\mathbf{a}_{i,2}^T \mathbf{x} + b_{i,2})}{\mathbf{c}_i^T \mathbf{x} + d_i}, \\ &4. \frac{-(\mathbf{a}_{i,1}^T \mathbf{x} + b_{i,1}) - (\mathbf{a}_{i,2}^T \mathbf{x} + b_{i,2})}{\mathbf{c}_i^T \mathbf{x} + d_i}, \end{aligned} \tag{15}$$

thus is a linear fractional function of $\mathbf{x}$ in the form of

$$r_i(\mathbf{x}) = \frac{\mathbf{q}_i^T \mathbf{x} + v_i}{\mathbf{c}_i^T \mathbf{x} + d_i}, \tag{16}$$

where $\mathbf{q}_i^T \in \mathbb{R}^3$ and $v_i \in \mathbb{R}$ is determined by which term in (15) is largest. Therefore, the gradient of $r_i(\mathbf{x})$ when $p = 1$ is

$$\nabla r_i(\mathbf{x}) = \frac{(\mathbf{c}_i^T \mathbf{x} + d_i)\mathbf{q}_i - (\mathbf{q}_i^T \mathbf{x} + v_i)\mathbf{c}_i}{(\mathbf{c}_i^T \mathbf{x} + d_i)^2} \tag{17}$$

2.3. Gradient of residual function when $p = \infty$

When $p = \infty$, the residual function (2) becomes

$$r_i(\mathbf{x}) = \frac{\max(|\mathbf{a}_{i,1}^T \mathbf{x} + b_{i,1}|, |\mathbf{a}_{i,2}^T \mathbf{x} + b_{i,2}|)}{\mathbf{c}_i^T \mathbf{x} + d_i}, \tag{18}$$

which, similar to (15), is the maximum over four terms:

$$r_i(\mathbf{x}) = \max(\frac{\mathbf{a}_{i,1}^T \mathbf{x} + b_{i,1}}{\mathbf{c}_i^T \mathbf{x} + d_i}, \frac{-\mathbf{a}_{i,1}^T \mathbf{x} + b_{i,1}}{\mathbf{c}_i^T \mathbf{x} + d_i}, \frac{\mathbf{a}_{i,2}^T \mathbf{x} + b_{i,2}}{\mathbf{c}_i^T \mathbf{x} + d_i}, \frac{-\mathbf{a}_{i,2}^T \mathbf{x} + b_{i,2}}{\mathbf{c}_i^T \mathbf{x} + d_i}), \tag{19}$$

thus is a linear fractional function of $\mathbf{x}$ in the same form with (16), and the gradient of $r_i(\mathbf{x})$ when $p = \infty$ is therefore formulated by (17) as well.

3. Proof of Lemma 2

Lemma 2 says:"Given a descent direction $\boldsymbol{\lambda}$ for a function $f(\mathbf{x})$ at $\hat{\mathbf{x}}$, the larger $\langle \boldsymbol{\lambda}, -\nabla f(\hat{\mathbf{x}}) \rangle$ is, the faster $f(\mathbf{x})$ is reduced along the direction $\hat{\mathbf{x}} + \alpha \boldsymbol{\lambda}$ for $\alpha > 0$".

Proof. By the definition of gradient, for a positive α,

$$\nabla f(x) = \lim_{\alpha \to 0^+} \frac{f(x + \alpha \boldsymbol{\lambda}) - f(x)}{\alpha \boldsymbol{\lambda}}, \tag{20}$$

such that

$$-\nabla f(x)^T \boldsymbol{\lambda} = \lim_{\alpha \to 0^+} \frac{f(x + \alpha \boldsymbol{\lambda}) - f(x)}{\alpha}. \tag{21}$$

If we have 2 descent directions $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ that satisfy

$$-\nabla f(x)^T \boldsymbol{\lambda}_1 > -\nabla f(x)^T \boldsymbol{\lambda}_2, \tag{22}$$

then

$$\begin{aligned}
&\nabla f(x)^T \boldsymbol{\lambda}_2 - \nabla f(x)^T \boldsymbol{\lambda}_1 \\
&= \lim_{\alpha \to 0^+} \frac{f(x + \alpha \boldsymbol{\lambda}_2) - f(x)}{\alpha} - \lim_{\alpha \to 0^+} \frac{f(x + \alpha \boldsymbol{\lambda}_1) - f(x)}{\alpha} \\
&= \lim_{\alpha \to 0^+} \frac{f(x + \alpha \boldsymbol{\lambda}_2) - f(x + \alpha \boldsymbol{\lambda}_1)}{\alpha} > 0,
\end{aligned} \tag{23}$$

which yields the conclusion

$$f(x + \alpha \boldsymbol{\lambda}_1) < f(x + \alpha \boldsymbol{\lambda}_2), \tag{24}$$

for any sufficiently small positive α. $\square$