Rotation Averaging and Strong Duality - Supplementary Material

Anders Eriksson[1], Carl Olsson[2,3], Fredrik Kahl[2,3] and Tat-Jun Chin[4]

[1]School of Electrical Engineering and Computer Science, Queensland University of Technology
[2]Department of Electrical Engineering, Chalmers University of Technology
[3]Centre for Mathematical Sciences, Lund University
[4]School of Computer Science, The University of Adelaide

Proof of Theorem 4.2

Theorem 4.2. *Let R_i^*, $i = 1,\dots,n$ denote a stationary point to the primal problem (P) for a cycle graph with n vertices. Let α_{ij} denote the angular residuals, i.e., $\alpha_{ij} = \angle(R_i^*\tilde{R}_{ij}, R_j^*)$. Then, R_i^*, $i = 1,\dots,n$ will be globally optimal and strong duality will hold for (P) if*

$$|\alpha_{ij}| \leq \frac{\pi}{n} \quad \forall (i,j) \in E.$$

Proof. A sufficient condition for strong duality to hold is that $\Lambda^* - \tilde{R} \succeq 0$ (Lemma 3.2), which is equivalent to $D_{R^*}(\Lambda^* - \tilde{R})D_{R^*}^T \succeq 0$ with the same notation and argument as in (23) and (24). For a cycle graph, we get $D_{R^*}(\Lambda^* - \tilde{R})D_{R^*}^T =$

$$\begin{bmatrix} \mathcal{E}_{12}+\mathcal{E}_{1n} & -\mathcal{E}_{12} & & & -\mathcal{E}_{1n} \\ -\mathcal{E}_{12}^T & \mathcal{E}_{12}^T+\mathcal{E}_{23} & -\mathcal{E}_{23} & & \\ & -\mathcal{E}_{23}^T & \ddots & \ddots & \\ & & \ddots & \ddots & \\ -\mathcal{E}_{1n}^T & & & & \end{bmatrix}. \tag{48}$$

As this matrix is symmetric, it implies for the first diagonal block that $\mathcal{E}_{12} - \mathcal{E}_{12}^T = \mathcal{E}_{1n}^T - \mathcal{E}_{1n}$. As all $\mathcal{E}_{ij} \in \mathrm{SO}(3)$, it follows that $\mathcal{E}_{12} = \mathcal{E}_{1n}^T = \mathcal{E}$ for some rotation $\mathcal{E} \in \mathrm{SO}(3)$. Similarly, for the second diagonal block $\mathcal{E}_{12} = \mathcal{E}_{23}^T = \mathcal{E}$ and by induction, the matrix $D_{R^*}(\Lambda^* - \tilde{R})D_{R^*}^T$ has the following tridiagonal (Laplacian-like) structure

$$\begin{bmatrix} \mathcal{E}+\mathcal{E}^T & -\mathcal{E} & & & -\mathcal{E}^T \\ -\mathcal{E}^T & \mathcal{E}+\mathcal{E}^T & -\mathcal{E} & & \\ & -\mathcal{E}^T & \ddots & \ddots & \\ & & \ddots & \ddots & -\mathcal{E} \\ -\mathcal{E} & & & -\mathcal{E}^T & \mathcal{E}+\mathcal{E}^T \end{bmatrix}. \tag{49}$$

Note that this means that the total error is equally distributed in an optimal solution among all the residuals, in particular, $\alpha_{ij} = \alpha$ for all $(i,j) \in E$, where α is the residual rotation angle of $\mathcal{E}$.

Let v denote the rotation axis of $\mathcal{E}$ and let u and w be an orthogonal base which is orthogonal to v. Then, define the two vectors $v_\pm = (\, v_{\pm,1} \;\; v_{\pm,2} \;\; \dots \;\; v_{\pm,n} \,)^T$, where $v_{\pm,i} = \cos(\frac{2\pi i}{n})u \pm \sin(\frac{2\pi i}{n})w$ for $i = 1,\dots,n$. Now it is straight-forward to check that $v_\pm$ are eigenvectors to (49) with eigenvalues $4\sin(\frac{\pi}{n} \pm \alpha)\sin(\frac{\pi}{n})$. The sign of the smallest of these two eigenvalues determines the positive definiteness of the matrix in (49). In other words, we have shown that if $|\alpha| \leq \frac{\pi}{n}$ then $D_{R^*}(\Lambda^* - \tilde{R})D_{R^*}^T \succeq 0$. □

Proof of Lemma 5.1

Lemma 5.1. *Let B be a positive semidefinite matrix. Then, the solution to (46) is given by,*

$$S^* = -BW\left[\left(W^TBW\right)^{\frac{1}{2}}\right]^\dagger. \tag{50}$$

Proof. From the Schur complement, we have that the 2×2 block matrix in (46) is positive semidefinite if and only if

$$I - S^TB^\dagger S \succeq 0, \tag{51}$$

$$(I - BB^\dagger)S = 0. \tag{52}$$

Hence the problem (46) is equivalent to

$$\min_{S\in\mathbb{R}^{3n\times 3}} \quad < W, S > \tag{53a}$$

$$\text{s.t.} \quad I - S^TB^\dagger S \succeq 0, \tag{53b}$$

$$(I - BB^\dagger)S = 0. \tag{53c}$$

The KKT conditions for (53), with Lagrangian multipliers Γ and Υ , become

$$W + 2B^\dagger S\Gamma + (I - BB^\dagger)\Upsilon = 0, \tag{54}$$

$$I - S^TB^\dagger S \succeq 0, \tag{55}$$

$$(I - BB^\dagger)S = 0, \tag{56}$$

$$\Gamma \succeq 0, \tag{57}$$

$$(I - S^TB^\dagger S)\Gamma = 0. \tag{58}$$

Rewrite (54) and (58) as

$$B^\dagger S\Gamma = -\frac{1}{2}W - \frac{1}{2}(I - BB^\dagger)\Upsilon, \tag{59}$$

$$\Gamma^T\Gamma = \Gamma^TS^TB^\dagger S\Gamma. \tag{60}$$

Since the pseudoinverse fulfills $B^\dagger B B^\dagger = B^\dagger$, combining (59) and (60) we obtain

$$\Gamma^2 = \Gamma^T S^T B^\dagger B B^\dagger S \Gamma = \tag{61}$$

$$= \frac{1}{4}\left(W + (I - BB^\dagger)\Upsilon\right)^T B \left(W + (I - BB^\dagger)\Upsilon\right) = \tag{62}$$

$$= \frac{1}{4} W^T B W. \tag{63}$$

Here the last equality follows since $B(I - BB^\dagger) = 0$. This gives

$$\Gamma = \frac{1}{2}\left(W^T B W\right)^{\frac{1}{2}}. \tag{64}$$

Inserting (64) in (59)

$$B^\dagger S \left(W^T B W\right)^{\frac{1}{2}} = -W - (I - BB^\dagger)\Upsilon, \tag{65}$$

$$\tag{66}$$

multiplying with B form the left on both sides and using (56), $BB^\dagger S = S$, we arrive at

$$S\left(W^T B W\right)^{\frac{1}{2}} = -BW, \tag{67}$$

and consequently

$$S = -BW\left[\left(W^T B W\right)^{\frac{1}{2}}\right]^\dagger. \tag{68}$$

Finally, since

$$\Gamma = \frac{1}{2}\left(W^T B W\right)^{\frac{1}{2}} \succeq 0, \tag{69}$$

$$I - S^T B^\dagger S =$$

$$= I - \left[\left(W^T B W\right)^{\frac{1}{2}}\right]^\dagger W^T B W \left[\left(W^T B W\right)^{\frac{1}{2}}\right]^\dagger \succeq 0, \tag{70}$$

the conditions (55) and (57) are satisfied then (50) must be a feasible and optimal solution to (53) and consequently also to (46). □